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Some Geometric properties in Orlicz- Cesaro Spaces

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ABSTRACT: On the Orlicz- Cesaro sequence spaces (ces_Φ) which are defined by using Orlicz function Φ , we show that the space ces_Φ equipped with both Amemiya and Luxemburg norms possesses uniform Opial property and uniform Kadec-Klee property if Φ satisfy the δ_2 -condition.

KEYWORDS: Orlicz-Cesaro sequence spaces, uniform Kadec-Klee property, uniform Opial property, Amemiya norm, Luxemburg norm.

INTRODUCTION

In the whole paper $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural numbers and of real numbers, respectively. The space of all real sequences is denoted by l^0 . Let $(X,\|\cdot\|)$ be a real normed space and $B(X)(S(X))$ be the closed unit ball (the unit sphere) of X.

A Banach space $(X,\|\cdot\|)$ which is a subspace of l^0 is said to be a *Kothe sequence space*, if :

(i) for any $x \in l^0$ and $y \in X$ such that $|x(i)| \le |y(i)|$ for all $i \in \mathbb{N}$, we have $x \in X$ and $\|x\| \le \|y\|$,

(ii) there is $x \in X$ with $x(i) \ne 0$ for all $i \in \mathbb{N}$

An element x from a Kothe sequence space X is called *order continuous* if for any sequence (x_n) in X_+ (the positive cone of X) such that $x_n \le |x|$ for all $n \in \mathbb{N}$ and $x_n \to 0$ coordinatewise, we have $\|x_n\| \to 0$.

A Kothe sequence space X is said to be *order continuous* if any $x \in X$ is order continuous. It is easy to see that $x \in X$ is order continuous if and only if $\|(0,0,\dots,0,x(n+1),x(n+2),\dots)\| \to 0$ as $n \to \infty$.

A Banach space X is said to have the *Kadec-Klee property* (or **H**-property) if every weakly convergent sequence on the unit sphere is convergent in norm.

Recall that a sequence $\{x_n\} \subset X$ is said to be *ε-separated sequence* for some $\varepsilon > 0$ if

$sep(x_n) = inf\{\|x_n - x_m\| : n \ne m\} > \varepsilon$.

A Banach space is said to have the *uniform Kadec-Klee property* (write (**UKK**) for short) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence (x_n) in $S(X)$ with $sep(x_n) > \varepsilon$ and $x_n \xrightarrow{w} x$, we have $\|x\| < 1 - \delta$. Every (**UKK**) Banach space has **H**-property (see [1])

The Opial property is important because Banach spaces with this property have the weak fixed point property (see [2]). Opial has proved in [3] that the sequence spaces $\ell_p (1 < p < \infty)$ have this condition but $L_p[0,2\pi] (p \ne 2,\ 1 < p < \infty)$ do not.

A Banach space X is said to have the *Opial property* (see [3]) if for any weakly null sequence (x_n) and every $x \ne 0$ in X, we have

$\lim_{n\to\infty} inf\|x_n\| < \lim_{n\to\infty} inf\|x_n + x\|$.

A Banach space X is said to have the *uniform Opial property* (see [4]) if for each $\varepsilon > 0$ there exists $\tau > 0$ such that for any weakly null sequence (x_n) in $S(X)$ and $x \in X$ with $\|x\| \ge \varepsilon$ the following inequality holds:

$1 + \tau \le \lim_{n\to\infty} inf\|x_n + x\|$.

For a real vector space X, a function $\mathfrak{M} : X \to [0,\infty]$ is called a *modular* if it satisfies the following conditions:

$(i)\ \mathfrak{M}(x) = 0$ if and only if $x = 0$,

$(ii)\ \mathfrak{M}(\alpha x) = \mathfrak{M}(x)$ for all scalar α with $|\alpha| = 1$,

$(iii)\ \mathfrak{M}(\alpha x + \beta y) \le \mathfrak{M}(x) + \mathfrak{M}(y)$, for all $x, y \in X$ and all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

The modular $\mathfrak{M}$ is called *convex* if

$(iii)\ \mathfrak{M}(\alpha x + \beta y) \le \alpha\mathfrak{M}(x) + \beta\mathfrak{M}(y)$, for all $x, y \in X$ and all $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

For any modular $\mathfrak{M}$ on X, the space

$X_{\mathfrak{M}} = \{x \in X : \mathfrak{M}(\lambda x) \to 0 \text{ as } \lambda \to 0\}$,

is called the *modular space*.

A sequence (x_n) of elements of $X_{\mathfrak{M}}$ is called *modular convergent* to $x \in X_{\mathfrak{M}}$ if there exists a $\lambda > 0$ such that $\mathfrak{M}(\lambda(x_n - x)) \to 0$, as $n \to \infty$.

If $\mathfrak{M}$ is a convex modular, the function

$\|x\| = inf\left\{\lambda > 0 : \mathfrak{M}\left(x/\lambda\right) \le 1\right\}$,

and

$\|x\|_A = \inf_{k>0} \frac{1}{k}\left(1 + \mathfrak{M}(kx)\right)$,

are two norms on $X_{\mathfrak{M}}$, which are called the *Luxemburg norm* and the *Amemiya norm*, respectively. In addition, $\|x\| \le \|x\|_A \le 2\|x\|$ for all $x \in X_{\mathfrak{M}}$ (see [5]).

Theorem 1.1 Let $(x_n) \subset X_{\mathfrak{M}}$ then $\|x_n\| \to 0$ (or equivalently $\|x_n\|_A \to 0$) if and only if $\mathfrak{M}(\lambda(x_n)) \to 0$, as $n \to \infty$, for every $\lambda > 0$.

Proof. See [6, Theorem 1.3(a)].

A modular $\mathfrak{M}$ is said to satisfy the Δ_2*-condition* ($\mathfrak{M} \in \Delta_2$) if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that

$\mathfrak{M}(2x) \leq K\mathfrak{M}(x) + \varepsilon$

for all $x \in X_{\mathfrak{M}}$ with $\mathfrak{M}(x) \leq a$.

If $\mathfrak{M}$ satisfies the Δ_2-condition for all $a > 0$ with $K \geq 2$ dependent on a, we say that $\mathfrak{M}$ satisfies the *strong* Δ_2*-condition* ($\mathfrak{M} \in \Delta_2^S$).

Theorem 1.2 Convergences in norm and in modular are equivalent in $X_{\mathfrak{M}}$ if $\mathfrak{M} \in \Delta_2$.

Proof. See [7, Lemma 2.3].

Theorem 1.3 If $\mathfrak{M} \in \Delta_2^S$ then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$|\mathfrak{M}(u+v) - \mathfrak{M}(u)| < \varepsilon$

whenever $u, v \in X_{\mathfrak{M}}$ with $\mathfrak{M}(u) \leq L$ and $\mathfrak{M}(v) \leq \delta$.

Proof. See [7, Lemma 2.1].

Theorem 1.4 If $\mathfrak{M} \in \Delta_2^S$, then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x\| \geq 1 + \delta$ whenever $\mathfrak{M}(x) \geq 1 + \varepsilon$.

Proof. See [7, Lemma 2.4].

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an *Orlicz function* if it is even, convex, continuous and vanishing at 0 and $\Phi(u) \to \infty$ as $u \to \infty$. Furthermore, we say that an Orlicz function Φ is an N'*-function* if $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. The *Orlicz sequence space*, ℓ_Φ, where Φ is an Orlicz function is defined as

$\ell_\Phi = \{x \in \ell^0 : \mathrm{I}_\Phi(\lambda x) < \infty \ \exists \lambda > 0\}$,

where $\mathrm{I}_\Phi(x) = \sum_{i=1}^{\infty} \Phi(x(i))$ is a convex modular on ℓ_Φ. Then ℓ_Φ is a Banach space with both Luxemburg norm $\|\cdot\|_{\ell_\Phi^L}$ and Amemiya norm $\|\cdot\|_{\ell_\Phi^A}$ (see [5]). Denoted by $K(x)$ the set of all $k > 0$ $k > 0$ such that $\|x\|_A = \frac{1}{k}(1 + \mathrm{I}_\Phi(kx))$, it is well known that $K(x) \neq \emptyset$ for all $x \in \ell_\Phi$ whenever Φ is an N'-function (see [8]).

An Orlicz function Φ is said to *satisfy the* δ_2*-condition* (we will write $\Phi \in \delta_2$ for short) if there exist constants $K \geq 2$ and $u_0 > 0$ such that the inequality $\Phi(2u) \leq K\Phi(u)$ holds for every $u \in \mathbb{R}$ satisfying $|u| \leq u_0$.

For $1 < p < \infty$, the Cesaro sequence space (write, ces_p, for short) is defined by

$$ces_p = \left\{ x \in \ell^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < \infty \right\},$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right)^{\frac{1}{p}} \qquad (1.1)$$

This space was first introduced by Shiue [9]. It is useful in the theory of Matrix operators and others (see [10] and [11]). Some geometric properties of the Cesaro sequence spaces ces_p were studied by many authors.

For an Orlicz function Φ the *Orlicz- Cesaro sequence space*, ces_Φ, is defined by

$ces_\Phi = \{x \in \ell^0 : \rho_\Phi(\lambda x) < \infty, \ \exists \lambda > 0\}$,

where

$\rho_\Phi(x) = \sum_{n=1}^{\infty} \Phi\left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)$,

is a convex modular on ces_Φ. The subspace E_Φ of ces_Φ is defined by

$E_\Phi = \{x \in \ell^0 : \rho_\Phi(\lambda x) < \infty, \ \forall \lambda > 0\}$.

It is worth noting that if $\Phi \in \delta_2$, then $\rho_\Phi \in \Delta_2^S$ and $ces_\Phi = E_\Phi$.

To simplify notations, we put $ces_\Phi^L = (ces_\Phi, \|\cdot\|_L)$ and $ces_\Phi^A = (ces_\Phi, \|\cdot\|_A)$. In the case when $\Phi(t) = |t|^p, (p > 1)$ the Orlicz- Cesaro sequence space ces_Φ becomes the Cesaro sequence space ces_p and the Luxemburg norm is that one defined by (1.1).

From now on, for $x \in \ell^0$ and $i \in \mathbb{N}$ we let

$x_{|_i} = (x(1), x(2), \ldots, x(i), 0, 0, \ldots)$,

$x_{|_{\mathbb{N}-i}} = (0, 0, \ldots, x(i+1), x(i+2), x(i+3), \ldots)$,

and

$supp\, x = \{i \in \mathbb{N} : x(i) \neq 0\}$.

Results

We first give an important fact for $\|x\|_A$ on ces_Φ^A.

Lemma 2.1 If Φ is an N'-function, then for each $x \in ces_\Phi^A$ there exists $k \in \mathbb{R}$ such that

$\|x\|_A = \frac{1}{k}(1 + \rho_\Phi(kx))$.

Proof. For each $x = (x(i))_{i=1}^{\infty} \in ces_\Phi$ we have $\bar{x} = \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)_{n=1}^{\infty} \in \ell_\Phi$. Observe that $\|x\|_{ces_\Phi^A} = \|\bar{x}\|_{\ell_\Phi^A}$, and Φ is an N'-function, by [8, Corollary 2.3] there exists $k \in \mathbb{R}$ such that

$$\|x\|_{ces_\Phi^A} = \|\bar{x}\|_{\ell_\Phi^A} = \frac{1}{k}(1 + \mathrm{I}_\Phi(k\bar{x}))$$
$$= \frac{1}{k}\left(1 + \sum_{n=1}^{\infty} \Phi\left(\frac{k}{n} \sum_{i=1}^{n} |x(i)| \right) \right) = \frac{1}{k}(1 + \rho_\Phi(kx)).$$

This completes the proof of our Lemma.

Proposition 2.2 Suppose that Φ is an N'-function and let $\{x_n\}$ be a bounded sequence in ces_Φ^A such that $x_n \xrightarrow{w} x$ for some $x \in ces_\Phi^A$. If $k_n \in K(x_n)$

and $k_n \to \infty$, then $x = 0$.

Proof. For each $n \in \mathbb{N}, \eta > 0$, put $G_{(n,\eta)} = \left\{ i \in \mathbb{N} : \frac{1}{i}\sum_{j=1}^{i} |x_n(j)| \ge \eta \right\}$. First, we claim that for each $\eta > 0, G_{(n,\eta)} = \varnothing$ for all large $n \in \mathbb{N}$. Otherwise, without loss of generality, we may assume that $G_{(n,\eta)} \ne \varnothing$ for all $n \in \mathbb{N}$ and for some $\eta > 0$. Then

$$\|x_n\|_A = \frac{1}{k_n}\left(1 + \rho_\Phi(k_n x_n)\right) \ge \frac{\Phi(k_n x_n)}{k_n} \qquad \left(i \in G_{(n,\eta)}\right).$$

By applying the assumption that Φ is an N'-function, we obtain $\|x_n\|_A \to \infty$, which contradicts to the fact that $\{x_n\}$ is bounded, hence we have the claim. By the claim, we have $\frac{1}{i}\sum_{j=1}^{i}|x_n(i)| \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$. This implies that $x_n(i) \to 0$ as $n \to \infty$ for all $i \in \mathbb{N}$. Since $x_n \xrightarrow{w} x$, we have $x_n(i) \to x(i)$ for all $i \in \mathbb{N}$, so it follows that $x(i) = 0$ for all $i \in \mathbb{N}$.

Lemma 2.3 For any Orlicz function Φ, we have $E_\Phi \subseteq \left\{ x \in ces_\Phi : \left\|x - x_{|_i}\right\|_A \to 0 \right\}$.

Proof. Write $A = \left\{ x \in ces_\Phi : \left\|x - x_{|_i}\right\|_A \to 0 \right\}$. Let $x \in E_\Phi$ and $\varepsilon > 0$. Since $x \in E_\Phi$, there exists $i_o \in \mathbb{N}$ such that $\rho_\Phi\left(\left(x - x_{|_i}\right)/\varepsilon\right) < \varepsilon$ for all $i > i_o$. Therefore, by the definition of $\|\cdot\|_A$ we have

$$\left\|\varepsilon^{-1}\left(x - x_{|_i}\right)\right\|_A \le 1 + \rho_\Phi\left(\left(x - x_{|_i}\right)/\varepsilon\right) < 1 + \varepsilon$$

for all $i > i_o$. This yields $\left\|\left(x - x_{|_i}\right)\right\|_A \to 0$ as $i \to \infty$ since ε is arbitrary. Hence $x \in A$, proving the Lemma.

Theorem 2.4 The space ces_Φ^A is **(UKK)** if Φ is an N'-function which satisfies the δ_2-condition.

Proof. For a given $\varepsilon > 0$, by Theorem 1.2 there exists $\delta \in (0,1)$ such that $\|y\|_A \ge \varepsilon/4$ implies $\rho_\Phi(y) \ge 2\delta$. Given $x_n \in B\left(ces_\Phi^A\right), x_n \to x$ weakly and $\|x_n - x_m\|_A \ge \varepsilon (n \ne m)$, we shall complete the proof by showing that $\|x\|_A \le 1 - \delta$. Indeed, if $x = 0$, then it is clear. So, we assume $x \ne 0$. In this case, by Proposition 2.2 we have that $\{k_n\}$ is bounded, where $k_n \in K(x_n)$. Passing to a subsequence if necessary we may assume that $k_n \to k$ for some $k > 0$. Since $\Phi \in \delta_2$, Lemma 2.3 assures that there exists $j \in \mathbb{N}$ such that $\left\|x_{|_j}\right\|_A \ge \|x\|_A - \delta$. Since the weak convergence of $\{x_n\}$ implies that $x_n \to x$ coordinatewise, we deduce that $x_n(i) \to x(i)$ uniformly on $\{1,2,\ldots,j\}$. Consequently, there exists $n_o \in \mathbb{N}$ such that

$$\left\|\left(x_n - x_m\right)_{|_j}\right\|_A \le \varepsilon/2 \quad \text{for all } n,m \ge n_o,$$

which implies

$$\left\|\left(x_n - x_m\right)_{|_{\mathbb{N}-j}}\right\|_A \ge \varepsilon/2 \quad \text{for all } n,m \ge n_o, m \ne n.$$

This gives or $\left\|x_{m_{|_{\mathbb{N}-j}}}\right\|_A \ge \varepsilon/4$ for all $n,m \ge n_o, m \ne n$, which yields $\left\|x_{n_{|_{\mathbb{N}-j}}}\right\|_A \ge \varepsilon/4$ for infinitely many $n \in \mathbb{N}$, hence $\rho_\Phi\left(x_{n|_{\mathbb{N}-j}}\right) \ge 2\delta$. Without loss of generality we may assume that $\left\|x_{n_{|_{\mathbb{N}-j}}}\right\|_A \ge \varepsilon/4$, for all $n \in \mathbb{N}$. By using the convexity of Φ and the inequality $\Phi(a+b) \ge \Phi(a) + \Phi(b)$, $a,b \in \mathbb{R}^+$ together with the fact that $k_n \ge 1$, we have

$$\begin{aligned}
1 - 2\delta &\ge \|x_n\|_A - \rho_\Phi\left(x_{n|_{\mathbb{N}-j}}\right) \\
&\ge \|x_n\|_A - \frac{1}{k_n}\rho_\Phi\left(k_n x_{n|_{\mathbb{N}-j}}\right) \\
&= \frac{1}{k_n} + \frac{1}{k_n}\sum_{i=1}^{\infty}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right) - \frac{1}{k_n}\sum_{i=j+1}^{\infty}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{i-j}|x_n(j+r)|\right) \\
&= \frac{1}{k_n} + \frac{1}{k_n}\sum_{i=1}^{j}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right) + \frac{1}{k_n}\left[\sum_{i=j+1}^{\infty}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right) - \sum_{i=j+1}^{\infty}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{i-j}|x_n(j+r)|\right)\right] \\
&= \frac{1}{k_n} + \frac{1}{k_n}\sum_{i=1}^{j}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right) + \frac{1}{k_n}\left[\sum_{i=j+1}^{\infty}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{j}|x_n(r)| + \frac{k_n}{i}\sum_{r=1}^{i-j}|x_n(j+r)|\right) - \sum_{i=j+1}^{\infty}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{i-j}|x_n(j+r)|\right)\right] \\
&\ge \frac{1}{k_n} + \frac{1}{k_n}\sum_{i=1}^{j}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{i}|x_n(r)|\right) + \frac{1}{k_n}\sum_{i=j+1}^{\infty}\Phi\left(\frac{k_n}{i}\sum_{r=1}^{j}|x_n(r)|\right) \\
&= \frac{1}{k_n} + \frac{1}{k_n}\rho_\Phi\left(k_n x_{n|_j}\right) \to \frac{1}{k_n} + \frac{1}{k_n}\rho_\Phi\left(k_n x_{|_j}\right) \ge \left\|x_{|_j}\right\|_A \ge \|x\|_A - \delta,
\end{aligned}$$

hence $\|x\|_A \le 1-\delta$.

Theorem 2.5 If Φ is an N'-function which satisfies δ-condition, then ces_Φ^A has the uniform opial property.

Proof. Take any $\varepsilon > 0$ and $x \in ces_\Phi^A$ with $\|x\|_A \ge \varepsilon$. Let (x_n) be weakly null sequence in $S(ces_\Phi^A)$. By $\Phi \in \delta_2$, and Theorem 1.2, there is $\xi \in (0,1)$ independent of x such that $\rho\left(\frac{x}{2}\right) > \xi$. Also, by $\Phi \in \delta_2$, we have $ces_\Phi^A = E_\Phi$. By Lemma 2.3, x is an order continuous element, this allows us to find $j_o \in \mathbb{N}$ such that

$$\left\|x_{|_{\mathbb{N}-j_o}}\right\|_A < \frac{\xi}{4}$$

and

$$\sum_{j=j_o+1}^{\infty} \Phi\left(\frac{1}{j}\sum_{i=1}^{j}\frac{|x(i)|}{2}\right) < \frac{\xi}{8}.$$

It follows that

$$\xi \le \sum_{j=1}^{j_o} \Phi\left(\frac{1}{j}\sum_{i=1}^{j}\frac{|x(i)|}{2}\right) + \sum_{j=j_o+1}^{\infty} \Phi\left(\frac{1}{j}\sum_{i=1}^{j}\frac{|x(i)|}{2}\right)$$

$$\le \sum_{j=1}^{j_o} \Phi\left(\frac{1}{j}\sum_{i=1}^{j}\frac{|x(i)|}{2}\right) + \frac{\xi}{8},$$

which implies

$$\frac{7\xi}{8} \le \sum_{j=1}^{j_o} \Phi\left(\frac{1}{j}\sum_{i=1}^{j}\frac{|x(i)|}{2}\right). \qquad (2.1)$$

From $x_n \xrightarrow{w} 0$, we have $x_n(i) \to 0$ for all $i \in \mathbb{N}$, which implies that $\rho_\Phi\left(x_{n|_{j_o}}\right) \to 0$. By Theorem 1.2 we have $\left\|x_{|_{j_o}}\right\|_A \to 0$, so there exists $n_o \in \mathbb{N}$ such that

$$\left\|x_{n|_{j_o}}\right\|_A < \frac{\xi}{4} \quad \text{for all } n > n_o.$$

Therefore,

$$\|x+x_n\|_A = \left\|(x+x_n)_{|_{j_o}} + (x+x_n)_{|_{\mathbb{N}-j_o}}\right\|_A$$

$$\ge \left\|x_{|_{j_o}} + x_{n|_{\mathbb{N}-j_o}}\right\|_A - \left\|x_{|_{\mathbb{N}-j_o}}\right\|_A - \left\|x_{n|_{j_o}}\right\|_A$$

$$\ge \left\|x_{|_{j_o}} + x_{n|_{\mathbb{N}-j_o}}\right\|_A - \frac{\xi}{2}. \qquad (2.2)$$

Since Φ is an N'-function, by Lemma 2.1 there exists $k_n > 0$ such that

$$\left\|x_{|_{j_o}} + x_{n|_{\mathbb{N}-j_o}}\right\|_A = \frac{1}{k_n}\left(1 + \rho_\Phi\left(k_n\left(x_{|_{j_o}} + x_{n|_{\mathbb{N}-j_o}}\right)\right)\right).$$

This together with (2.2) and the fact that $\rho_\Phi(y+z) \ge \rho_\Phi(y) + \rho_\Phi(z)$ if $supp\, y \cap supp\, z = \varnothing$, we have

$$\|x+x_n\|_A \ge \frac{1}{k_n} + \frac{1}{k_n}\rho_\Phi\left(k_n x_{|_{j_o}}\right) + \frac{1}{k_n}\rho_\Phi\left(k_n x_{n|_{\mathbb{N}-j_o}}\right) - \frac{\xi}{2}$$

$$\ge \left\|x_{n|_{\mathbb{N}-j_o}}\right\|_A + \frac{1}{k_n}\rho_\Phi\left(k_n x_{|_{j_o}}\right) - \frac{\xi}{2} \qquad (2.3)$$

We may assume without loss of generality that $k_n \ge \frac{1}{2}$. Since $2k_n \ge 1$, by convexity of Orlicz function Φ we have that $\rho_\Phi\left(k_n x_{|_{j_o}}\right) \ge 2k_n \rho_\Phi\left(x_{|_{j_o}}\right)$. Thus inequalities (2.1) and (2.3) imply that

$$\|x+x_n\|_A \ge \left\|x_{n|_{\mathbb{N}-j_o}}\right\|_A + 2\rho_\Phi\left(\frac{x_{|_{j_o}}}{2}\right) - \frac{\xi}{2}$$

$$> \left\|x_{n|_{\mathbb{N}-j_o}}\right\|_A + 2\sum_{j=1}^{j_o}\Phi\left(\frac{1}{j}\sum_{i=1}^{j}\frac{|x(i)|}{2}\right) - \frac{\xi}{2}$$

$$> 1 - \frac{\xi}{4} + \frac{14\xi}{8} - \frac{\xi}{2}$$

$$= 1 + \xi \qquad \text{for all } n > n_o,$$

which deduces $\liminf_{n\to\infty}\|x+x_n\|_A \ge 1+\xi$.

Theorem 2.6 If Φ is an Orlicz function which satisfies δ_2-condition, then ces_Φ^L has the uniform opial property.

Proof. Take any $\varepsilon > 0$ and $x \in ces_\Phi$ with $\|x\|_L \ge \varepsilon$. Let (x_n) be weakly null sequence in $S(ces_\Phi^L)$. By $\Phi \in \delta_2$, we have $\rho_\Phi \in \Delta_2^S$. Thus by Theorem 1.2, there is $\eta \in (0,1)$ independent of x such that $\eta < \rho_\Phi(x) < \infty$. Also, by $\rho_\Phi \in \Delta_2^S$, Theorem 1.3 asserts that there exists $\eta_1 \in (0,\eta)$ such that

$$|\rho_\Phi(y+z) - \rho_\Phi(y)| < \frac{\eta}{4}, \qquad (2.4)$$

whenever $\rho_\Phi(y) \le 1$ and $\rho_\Phi(z) \le \eta_1$.

Since $\rho_\Phi(x) < \infty$, we choose $j_o \in \mathbb{N}$ such that

$$\sum_{j=j_o+1}^{\infty}\Phi\left(\frac{1}{j}\sum_{i=j_o+1}^{j}|x(i)|\right) < \sum_{j=j_o+1}^{\infty}\Phi\left(\frac{1}{j}\sum_{i=1}^{j}|x(i)|\right) < \frac{\eta_1}{4}. \qquad (2.5)$$

This gives

$$\eta < \sum_{j=1}^{j_o}\Phi\left(\frac{1}{j}\sum_{i=1}^{j}|x(i)|\right) + \sum_{j=j_o+1}^{\infty}\Phi\left(\frac{1}{j}\sum_{i=1}^{j}|x(i)|\right)$$

$$\le \sum_{j=1}^{j_o}\Phi\left(\frac{1}{j}\sum_{i=1}^{j}|x(i)|\right) + \frac{\eta_1}{4},$$

which implies

$$\sum_{j=1}^{j_o}\Phi\left(\frac{1}{j}\sum_{i=1}^{j}|x(i)|\right) > \eta - \frac{\eta_1}{4} > \eta - \frac{\eta}{4} = \frac{3\eta}{4}.$$

This together with the assumption that $x_n \xrightarrow{w} 0$, there exists $n_o \in \mathbb{N}$ such that

$$\frac{3\eta}{4} \le \sum_{j=1}^{j_o}\Phi\left(\frac{1}{j}\sum_{i=1}^{j}|x_n(i) + x(i)|\right). \qquad (2.6)$$

for all $n > n_o$, since the weak convergence implies the coordinatewise convergence. Again by $x_n \xrightarrow{w} 0$, there exists $n_1 > n_o$ such that $\rho_\Phi\left(x_{n|_{j_o}}\right) < \eta_1$ for all $n > n_1$, so from (2.4) we obtain

$$\left|\rho_\Phi\left(x_{n|_{\mathbb{N}-j_o}} + x_{n|_{j_o}}\right) - \rho_\Phi\left(x_{n|_{\mathbb{N}-j_o}}\right)\right| < \frac{\eta}{4},$$

since $\rho_\Phi(x_n) = 1$. Hence,

$$1 - \frac{\eta}{4} = \rho_\Phi(x_n) - \frac{\eta}{4} < \rho_\Phi\left(x_{n|_{\mathbb{N}-j_o}}\right) = \sum_{j=j_o+1}^{\infty}\Phi\left(\frac{1}{j}\sum_{i=j_o+1}^{j}|x_n(i)|\right),$$

for all $n > n_1$. This together with (2.4), (2.5) and (2.6) imply that for any $n > n_1$,

$$\begin{aligned}\rho_\Phi(x_n + x) &= \sum_{j=1}^{j_0} \Phi\left(\frac{1}{j}\sum_{i=1}^{j}|x_n(i)+x(i)|\right) + \sum_{j=j_0+1}^{\infty} \Phi\left(\frac{1}{j}\sum_{i=1}^{j}|x_n(i)+x(i)|\right)\\ &> \sum_{j=1}^{j_0} \Phi\left(\frac{1}{j}\sum_{i=1}^{j}|x_n(i)+x(i)|\right) + \sum_{j=j_0+1}^{\infty} \Phi\left(\frac{1}{j}\sum_{i=j_0+1}^{j}|x_n(i)+x(i)|\right)\\ &\geq \frac{3\eta}{4} + \sum_{j=j_0+1}^{\infty} \Phi\left(\frac{1}{j}\sum_{i=j_0+1}^{j}|x_n(i)|\right) - \frac{\eta}{4}\\ &\geq \frac{3\eta}{4} + \left(1-\frac{\eta}{4}\right) - \frac{\eta}{4} = 1 + \frac{\eta}{4}.\end{aligned}$$

By $\rho_\Phi \in \Delta_2^S$, and by Theorem 1.4, there is τ depending on η only such that $\|x_n + x\|_L \geq 1 + \tau$.

Corollary 2.7 ([12, Theorem 2]) For any $1 < p < \infty$, the space ces_p has the uniform Opial property.

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References

1. Huff R (1985) Banach spaces which are nearly uniformly convex. *Rocky Mountain J. Math.* **10**, 743-9.
2. Gossez JP and Lami Dozo E (1972) Some geometric properties related to fixed point theory for nonexpensive mappings. *Pacific J. Math.* **40**, 565-73.
3. Opial Z (1967) Weak convergence of the sequence of successive approximations for non expensive mappings. *Bull. Amer. Math. Soc.* **73**, 591-7.
4. Prus S (1992) Banach spaces with uniform Opial property. *Math. Chronicle, New Zealand* **18(8)**, 697-704.
5. Musielak J (1983) *Orlicz spaces and modular spaces*. Lecture Notes in Math. 1034, Springer-Verlag.
6. Maligranda L (1985) *Orlicz spaces and Interpolation*. Seminars in Mathematics, Campinas Vol. 5.
7. Cui YA and Hudzik H (1998) On the uniform opial property in some modular sequence spaces. *Funct. Approx. Comment.* **26,** 93-102.
8. Cui YA , Hudzik H, Nowak M and Pluciennik R (1999) Some geometric properties in Orlicz sequence spaces equipped with Orlicz norm. *J. Convex Anal*. **6(1)**, 91-113.
9. Shiue JS (1970) Cesaro sequence spaces. *Tamkang J. Math.* **1**, 143-50.
10. Lee PY (1984) Cesaro sequence spaces. *Math. Chronicle, New Zealand* **13**, 29-45.
11. Liu YQ, Wu BE and Lee PY (1996) *Method of sequence spaces*. Guangdong of Science and Technology Press (in Chinese).
12. Cui YA and Hudzik H (1999) Some geometric properties related to fixed point theory in Cesaro spaces. *Collect. Math.* **50(3)** , 277-88.