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Coefficients and Exponents of Power Series Representing Rational Functions

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Abstract Using well-known results from the theory of uniform distribution modulo 1, three criteria for rationality of power series based on the shape of their coefficients and their exponents are derived. The results improve those established earlier by M Newman.

KEYWORDS and PHRASES: power series, rational functions, uniform distribution modulo 1.

Introduction

In 1960, M Newman[1] proved the following remarkable assertions using some equally unexpected results from the theory of uniform distribution modulo 1:

I. Let á be a real number. Let g be a polynomial of degree ≥ 1 Define $G(z) = \sum_{n=0}^{\infty} g([n\alpha])z^n$. Then $G(z)$ is a rational function of $z \Leftrightarrow$ á is rational.

Here and throughout $[x]$ denotes the integral part of the real number x.

II. Suppose that á > 0. Define $F(z) = \sum_{n=0}^{\infty} z^{[n\alpha]}$. Then $F(z)$ is a rational function of $z \Leftrightarrow$ á is rational.

Soon after, a number of extensions to Newman's theorems were derived, *eg* Mordell[2], using complex analytic method, showed that if $g(x,y)$ is a polynomial in two variables with real coefficients, of degree ≥1 in x, then $\sum_{n=0}^{\infty} g([n\alpha], n\alpha)z^n$ is a rational function of $z \Leftrightarrow$ á is rational. Schwarz[3] proved that if f is a polynomial with *rational* coefficients of degree $m \geq 1$, p is a non-constant polynomial with real coefficients, and $\sum_{n=0}^{\infty} f([p(n)])z^n$ is a rational function of z, then either $p(x) - p(0)$ has only rational coefficients or $p(x)$ is of the form $p(0)$ + á $\sum_{i=1}^{r} a_i x^i$,where $\acute{a}^m$ is rational and the a_i are rational. Meijer[4] proved that if f is a non-constant polynomial with *complex* coefficients, á is a real number, and $k \geq 1$, then $\sum_{n=0}^{\infty} f([\alpha n^k])z^n$ is a rational function of $z \Leftrightarrow$ á is rational. Meijer's proof made use of divided differences as well as results from the theory of uniform distribution of a system of arithmetic functions. Cantor[5,6] showed that if p is a polynomial with real coefficients, then $\sum_{n=0}^{\infty} [p(n)]z^n$ is a rational function of $z \Leftrightarrow$ all coefficients of p, except perhaps the constant term, are rational. Our objective here is, as with Newman, to use results from the theory of uniform distribution modulo *1* to derive extensions of the above-mentioned results of Newman not previously covered. Our three principal theorems read as follows:

Theorem 1. Let á and â be real numbers, and let g be a polynomial over **C** of positive degree. Define $G(z) := \sum_{n=0}^{\infty} g([n\alpha + \beta])z^n$. Then $G(z)$ is a rational function of $z \Leftrightarrow$ á is rational.

Theorem 2. Let á > 0, â be real numbers, and let $F(z) = \sum_{n=1}^{\infty} z^{[n\alpha+\beta]}$. Then $F(z)$ is a rational function of $z \Leftrightarrow$ á is rational.

Theorem 3. Let á and â be real numbers, and let f, g be polynomials over **C** of positive degrees. Assuming that $g([n\alpha + \beta]) \neq 0$ for each non-negative integer n, define $G(z) := \sum_{n=0}^{\infty} \left(\frac{f}{g}\right)([n\alpha + \beta])z^n$. If $G(z)$ is a rational function of z, then á is rational.

An auxiliary lemma

We first prove a lemma which constitutes the crux of the proofs of our three theorems.

Lemma 1. Let α be a real irrational number, β be a real number, and S be a finite set of non-integral real numbers. Then there are infinitely many positive integers m such that

$[\{m\alpha + \beta\} + \gamma] = [\gamma]$ for all $\gamma \in S$,

and infinitely many positive integers n such that

$[\{n\alpha + \beta\} + \gamma] = 1 + [\gamma]$ for all $\gamma \in S$.

Here and throughout $\{x\}$ denotes the fractional part of the real number x, so that

$x = [x] + \{x\}$.

Proof. Observe that the two assertions are, respectively, equivalent to

$0 \le \{m\alpha + \beta\} + \{\gamma\} < 1$ for all $\gamma \in S$,

and

$0 \le \{n\alpha + \beta\} + \{\gamma\} - 1 < 1$ for all $\gamma \in S$.

They follow immediately from the fact that the sequence $(\{n\alpha+\beta\})_{n=1}^{\infty}$ is everywhere dense in [0,1). By example 2.1, p. 8 of Kuipers and Niederreiter[7], the sequence $(n\alpha)_{n=1}^{\infty}$ is uniformly distributed modulo 1. Thus by Lemma 1.1, p. 3 of Kuipers and Niederreiter[7], the sequence $(n\alpha+\beta)_{n=1}^{\infty}$ is uniformly distributed modulo *1* and finally by exercise 1.6, p. 6 of Kuipers and Niederreiter[7], the sequence $(\{n\alpha+\beta\})_{n=1}^{\infty}$ is everywhere dense in [0,1) as desired.

Proof of Theorem 1

Let α be irrational and assume that $G(z)$ is a rational function of z. Then there are polynomials $A(z)$ and $B(z)$ of degrees $a \ge 1$, and b, respectively, such that $G(z) = \dfrac{B(z)}{A(z)}$. Without loss of generality, let $A(z) := z^a - c_1 z^{a-1} - \ldots - c_{a-1} z - c_a$. From $A(z)G(z) = B(z)$, equating the corresponding coefficients of z^{n+a}, we get

(3.1) $$g([n\alpha+\beta]) = \sum_{r=1}^{a} g([n\alpha+\beta+r\alpha])c_r$$

for $n \ge \max(0, b-a+1)$.

Since g is a polynomial of degree (say) $p \ge 1$, then

$$\lim_{n\to\infty} \frac{g([n\alpha+\beta+r\alpha])}{g([n\alpha+\beta])} = \lim_{n\to\infty} \frac{[n\alpha+\beta+r\alpha]^p}{[n\alpha+\beta]^p} = 1,$$

so that (3.1) implies

(3.2) $c_1 + c_2 + \ldots + c_a = 1$.

Therefore, (3.1) and (3.2) together give

(3.3) $$\sum_{r=1}^{a} (g([n\alpha+\beta+r\alpha]) - g([n\alpha+\beta]))c_r = 0.$$

Using $[n\alpha+\beta+r\alpha] = [\{n\alpha+\beta\}+r\alpha] + [n\alpha+\beta]$, we have

$$g([n\alpha+\beta+r\alpha]) - g([n\alpha+\beta]) = \sum_{k=1}^{p} \frac{g^{(k)}([n\alpha+\beta])}{k!}[\{n\alpha+\beta\}+r\alpha]^k.$$

Multiplying both sides of this equation by c_r, summing over r, and using (3.3), we get, for large n,

(3.4)

$$\sum_{r=1}^{a} [\{n\alpha+\beta\}+r\alpha]c_r + \sum_{r=1}^{a}\sum_{k=2}^{p} \frac{g^{(k)}([n\alpha+\beta])}{k!\,g'([n\alpha+\beta])}[\{n\alpha+\beta\}+r\alpha]^k c_r = 0.$$

For $p = 1$, the last sum on the left hand side of (3.4) is empty, while for $p \ge 2$, we have

$$\lim_{n\to\infty} \frac{g^{(k)}([n\alpha+\beta])}{g'([n\alpha+\beta])}[\{n\alpha+\beta\}+r\alpha]^k = 0,$$

when $2 \le k \le p$, $1 \le r \le a$. Thus

(3.5) $$\lim_{n\to\infty} \sum_{r=1}^{a} [\{n\alpha+\beta\}+r\alpha]c_r = 0.$$

The numbers $r\alpha$ in (3.5) are not integers. By Lemma 1, we can find integers m and n such that the expressions

$$\sum_{r=1}^{a} [\{m\alpha+\beta\}+r\alpha]c_r = \sum_{r=1}^{a} [r\alpha]c_r$$

and

$$\sum_{r=1}^{a} [\{n\alpha+\beta\}+r\alpha]c_r = \sum_{r=1}^{a} (1+[r\alpha])c_r$$

can be made arbitrarily small, which contradicts (3.2).

Now assume that α is rational. Set $\alpha = \frac{c}{d}$, where c and d are relatively prime integers with $d > 0$. Applying the division algorithm, we get $n = md + r$, with $0 \le r \le d-1$, and so $n\alpha + \beta = n\frac{c}{d} + \beta = (md+r)\frac{c}{d} + \beta = mc + \frac{rc}{d} + \beta$, so *that* $[n\alpha+\beta] = mc + [\frac{rc}{d}+\beta]$. Thus

$$G(z) = \sum_{n=0}^{\infty} g([n\alpha+\beta])z^n = \sum_{r=0}^{d-1}\sum_{m=0}^{\infty} g(mc + [\tfrac{rc}{d}+\beta])z^{md+r}$$

$$= \sum_{r=0}^{d-1}\sum_{m=0}^{\infty}\sum_{k=0}^{p} \frac{g^{(k)}\left(\left[\frac{rc}{d}+\beta\right]\right)}{k!}(mc)^k z^{md+r}$$

$$= \sum_{r=0}^{d-1}\sum_{k=0}^{p} \frac{g^{(k)}\left(\left[\frac{rc}{d}+\beta\right]\right)}{k!} c^k z^r \sum_{m=0}^{\infty} m^k z^{md}.$$

From $\sum_{m=0}^{\infty} m^k z^m = \left(z\frac{d}{dz}\right)^k \left(\frac{1}{1-z}\right)$ is rational, it follows that $G(z)$ is a rational function of z.

Proof of Theorem 2

Since α is a positive real number, then there is a positive integer t_0 such that $t\alpha + \beta \ge 0$ for all positive integers $t \ge t_0$. Thus

$$F(z) = \sum_{t=1}^{t_0-1} z^{[t\alpha+\beta]} + \sum_{t=t_0}^{\infty} z^{[t\alpha+\beta]}.$$

Now $F(z)$ is rational $\Leftrightarrow \sum_{t=t_0}^{\infty} z^{[t\alpha+\beta]}$ is rational. Without loss of generality, we may assume that $t\alpha + \beta \ge 0$ for all positive integers t.

Suppose that α is irrational. Let $X(n)$ denote the number of solutions of $n = [t\alpha+\beta]$ in positive integers t. Then $F(z) = \sum_{n=0}^{\infty} X(n)z^n$.

Case 1 : for each positive integer t, $t\alpha+\beta$ is not integral.

Let N be a non-negative integer $\geq \beta$. Then for $n \geq N$, $X(n)$ is the number of integers t satisfying $n - \beta \leq t\alpha < n + 1 - \beta$, and since for each positive integer t, $t\alpha + \beta$ is not integral, and $X(n) = [(n + 1- \beta)/\alpha] - [(n - \beta)/\alpha]$, then

$$F(z) = \sum_{n=0}^{N-1} X(n)z^n + \sum_{n=N}^{\infty}\left(\left[\frac{n+1-\beta}{\alpha}\right]-\left[\frac{n-\beta}{\alpha}\right]\right)z^n.$$

Case 2: there is a positive integer k with $k\alpha+ \beta = \ell$, where ℓ is a non-negative integer.

We have $\beta = \ell - k\alpha$. Thus for each positive integer t not equal to k, we have $t\alpha + \beta = t\alpha + \ell - k\alpha = (t-k)\alpha + \ell$ being non-integral. This implies that k is the only positive integer such that $k\alpha + \beta$ is integral. Now let M be a positive integer for which $M > max(\beta, k\alpha + \beta)$. Thus for $n \geq M$, $X(n)$ is the number of integers t satisfying $n - \beta < t\alpha < n + 1 - \beta$. From $X(n) = [(n+1-\beta)/\alpha] - [(n-\beta)/\alpha]$, we get

$$F(z) = \sum_{n=0}^{M-1} X(n)z^n + \sum_{n=M}^{\infty}\left(\left[\frac{n+1-\beta}{\alpha}\right]-\left[\frac{n-\beta}{\alpha}\right]\right)z^n.$$

From both cases, we conclude that

$$F(z) = \sum_{n=0}^{K-1} X(n)z^n + \sum_{n=K}^{\infty}\left(\left[\frac{n+1-\beta}{\alpha}\right]-\left[\frac{n-\beta}{\alpha}\right]\right)z^n$$

for some integer K. Now note that

$$\sum_{n=K}^{\infty}\left(\left[\frac{n+1-\beta}{\alpha}\right]-\left[\frac{n-\beta}{\alpha}\right]\right)z^n = \frac{1-z}{z}\sum_{n=K+1}^{\infty}\left[n\left(\frac{1}{\alpha}\right)-\frac{\beta}{\alpha}\right]z^n - \left[\frac{K-\beta}{\alpha}\right]z^K,$$

and so

$$F(z) = \sum_{n=0}^{K-1} X(n)z^n + \frac{1-z}{z}\left(\sum_{n=0}^{\infty}\left[\frac{n}{\alpha}-\frac{\beta}{\alpha}\right]z^n - \sum_{n=0}^{K}\left[\frac{n-\beta}{\alpha}\right]z^n\right) - \left[\frac{K-\beta}{\alpha}\right]z^K.$$

According to Theorem 1, $\sum_{n=0}^{\infty}\left[\frac{n}{\alpha}-\frac{\beta}{\alpha}\right]z^n$ is not a rational function of z, and so $F(z)$ is not a rational function of z

Now suppose that $\alpha = \frac{c}{d}$ is rational with relatively prime positive integers c and d. Using $t = md + r$, $0 \leq r \leq d - 1$, we have

$$\begin{aligned} z^{[\beta]} + F(z) &= \sum_{t=0}^{\infty} z^{[t\alpha+\beta]} = \sum_{r=0}^{d-1}\sum_{m=0}^{\infty} z^{[mc+\frac{rc}{d}+\beta]} &&= \sum_{r=0}^{d-1}\sum_{m=0}^{\infty} z^{mc+[\frac{rc}{d}+\beta]} \\ &= \sum_{r=0}^{d-1} z^{[\frac{rc}{d}+\beta]} \sum_{m=0}^{\infty} (z^c)^m &&= \sum_{r=0}^{d-1} z^{[\frac{rc}{d}+\beta]} \frac{1}{1-z^c} \end{aligned}$$

so that $F(z)$ is a rational function of z.

Proof of Theorem 3

Suppose that α is irrational but $G(z)$ is a rational function of z. Then there are polynomials $A(z)$ and $B(z)$ of degrees $a \geq 1$, and b, respectively, such that $G(z) = \frac{B(z)}{A(z)}$. Without loss of generality, let $A(z) = z^a - c_1 z^{a-1} - \dots - c_{a-1}z - c_a$. From $A(z)G(z) = B(z)$, it follows by equating the corresponding coefficients of z^{n+a}, when $n > a + b$, that

$$\left(\frac{f}{g}\right)([n\alpha+\beta]) = \sum_{r=1}^{a}\left(\frac{f}{g}\right)([n\alpha+\beta+r\alpha])c_r.$$

Since f and g are polynomials of positive degrees, then

$$(5.1) \qquad \lim_{n\to\infty}\frac{f([n\alpha+\beta+r\alpha])}{f([n\alpha+\beta])} = \lim_{n\to\infty}\frac{g([n\alpha+\beta])}{g([n\alpha+\beta+r\alpha])} = 1.$$

Thus

$$\lim_{n\to\infty}\frac{\left(\frac{f}{g}\right)([n\alpha+\beta+r\alpha])}{\left(\frac{f}{g}\right)([n\alpha+\beta])} = \lim_{n\to\infty}\frac{f([n\alpha+\beta+r\alpha])\,g([n\alpha+\beta])}{f([n\alpha+\beta])\,g([n\alpha+\beta+r\alpha])} = 1$$

for each $r = 1,\ldots,a$. Thus (5.1) yields

$$(5.2) \qquad c_1 + c_2 + \ldots + c_a = 1$$

and so

$$(5.3) \qquad \sum_{r=1}^{a}\left(\left(\frac{f}{g}\right)([n\alpha+\beta+r\alpha]) - \left(\frac{f}{g}\right)([n\alpha+\beta])\right)c_r = 0.$$

Since $[n\alpha+\beta+r\alpha] = [\{n\alpha+\beta\}+r\alpha]+[n\alpha+\beta]$, then by Taylor's theorem, for each large positive integer n, there is a real number $c_{n,r}$ lying between $[n\alpha+\beta]$ and $[n\alpha+\beta+r\alpha]$ such that

$$\left(\frac{f}{g}\right)([n\alpha+\beta+r\alpha]) - \left(\frac{f}{g}\right)([n\alpha+\beta]) = \frac{\left(\frac{f}{g}\right)'([n\alpha+\beta+r\alpha])}{1!}[\{n\alpha+\beta\}+r\alpha] + \frac{\left(\frac{f}{g}\right)''(c_{n,r})}{2!}[\{n\alpha+\beta\}+r\alpha]^2.$$

Multiplying both sides of this last equation by c_r, summing over r, and using (5.3), we get, for large n,

$$(5.4) \qquad \sum_{r=1}^{a}[\{n\alpha+\beta\}+r\alpha]c_r + \sum_{r=1}^{a}\frac{\left(\frac{f}{g}\right)''(c_{n,r})}{2!\left(\frac{f}{g}\right)'([n\alpha+\beta])}[\{n\alpha+\beta\}+r\alpha]^2 c_r = 0.$$

For each $r = 1,\ldots,a$, since $c_{n,r}$ lies between $[n\alpha+\beta]$ and $[n\alpha+\beta+r\alpha]$ which yields $[n\alpha+\beta+r\alpha]-[n\alpha+\beta] = [\{n\alpha+\beta\}+r\alpha]$ and $\frac{(f/g)''(z)}{(f/g)'(z)}$ is of the form $\frac{p(z)}{q(z)}$ where $deg\, p(z) \le 4j+\ell-1$, $deg\, q(z) = 4j+\ell$, $deg\, g(z) = j$, $deg\, (den(f/g)') = \ell$, where *den* denotes denominator, then $\lim_{n\to\infty}\frac{(f/g)''(c_{n,r})}{(f/g)'([n\alpha+\beta])} = 0$. Thus for $r = 1, 2\ldots,a$, we have $\lim_{n\to\infty}\frac{(f/g)''(c_{n,r})}{2!(f/g)'([n\alpha+\beta])}[\{n\alpha+\beta\}+r\alpha]^2 c_r = 0$, and so by (5.4)

$$(5.5) \qquad \lim_{n\to\infty}\sum_{r=1}^{a}[\{n\alpha+\beta\}+r\alpha]\,c_r = 0.$$

The numbers $r\alpha$ in (5.5) are not integers. Since the sequence $(n\alpha+\beta)_{n=1}^{\infty}$ is uniformly distributed modulo 1, by Lemma 1 we can find integers m and n such that the expressions $\sum_{r=1}^{a}[\{m\alpha+\beta\}+r\alpha])c_r = \sum_{r=1}^{a}[r\alpha]c_r$ and $\sum_{r=1}^{a}[\{n\alpha+\beta\}+r\alpha]c_r = \sum_{r=1}^{a}(1+[r\alpha])c_r$ can be made arbitrarily small, and this contradicts (5.2), which ends the proof.

Contrary to Theorems 1 and 2, the converse of Theorem 3 fails in general as witnessed from the following example. Take $\frac{f}{g}(z) = \frac{1}{z+1}$, $\alpha = 1, \beta = 0$. Then $\sum_{n\ge 0}\frac{f}{g}([n\alpha+\beta])z^n = \sum_{n\ge 0}\frac{z^n}{n+1} = -\frac{\log(1-z)}{z}$, which is certainly not a rational power series.

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