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On Simple Iterative Ordinary Differential Equations

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ABSTRACT One class of the iterative ordinary differential equations is the simple iterative ordinary differential equation. In this paper, the local existence and uniqueness results for the first order of degree m of simple iterative ordinary differential equations are proved. The global existence and uniqueness results for the first order of second, third, fourth and m^{th} degree are also proved.

KEYWORDS: iterative, differential, equations.

INTRODUCTION

The first order iterative ordinary differential equation of degree m in the closed interval [0, a] is of the form

$$y'(x) = f(x, y(x),\ y^2(x), y^3(x), \ldots,\ \ y^m(x)) \quad (1)$$

with the initial condition

$$y(0) = c, \qquad (2)$$

where c is a positive real number in $[0, a]$, m is a positive integer greater than 1 and

$$\begin{aligned} y^2(x) &= y(y(x)) \\ y^3(x) &= y(y(y(x))) = y(y^2(x)) \\ y^4(x) &= y(y^3(x)) \\ &\vdots \\ y^m(x) &= y(y^{m-1}(x)). \end{aligned}$$

Pelczar[1-3], introduced the first order iterative ordinary differential equations of second degree and Podisuk[4] worked on the first and higher order iterative ordinary differential equations of degree m as well as the iterative partial differential equations.

One of the special problems of the first order iterative ordinary differential equations is that of the first order simple iterative ordinary differential equation of degree m which has the form

$$y'(x) = y^m(x),\ x \in [0, a], \qquad (3)$$

with the initial condition

$$y(0) = c, \qquad (4)$$

where c is a positive real number in $[0, a]$ and m is a positive integer greater than 1.

LOCAL EXISTENCE AND UNIQUENESS RESULTS

By the solution of the problems (3)-(4), we mean a function $y \in C^1[0, a]$ satisfying (3) and (4) in the closed interval $[0, a]$. Thus the problems (3)-(4) are equivalent to the problem of finding a continuous solution of the integral equation

$$y(x) = c + \int_0^x y^m(t)\,dt \qquad (5)$$

Choose the Banach space $B = C[0, a]$ equipped with norm $|u| = \max_{x\varepsilon[0,a]} |u(x)|$. Set $S(\rho) = \{u \in B$: $0 \le u \le \rho$, $|u(x) - u(\bar{x})| \le M|x - \bar{x}|\}$ where $\rho = c + a^2$, M is the positive real number and $T_m = 1 + M + M^2 + M^3 + \ldots + M^{m-1}$.

Define $T : S(\rho) \to S(\rho)$ by $(Tu)(x) = c + \int_0^x u^m(t)\,dt$.

Theorem 1 Suppose $c + a^2 \le a$, $a \le M$ and $aT_m < 1$ then T has a unique fixed point, that is, there is a unique solution to the problems (3)-(4).

Proof We have $0 \le (Tu) \le c + \int_0^x |u^m(t)|\,dt \le c + a^2 = \rho$

and $|(Tu)(x) - (Tu)(\bar{x})| \le \int_{\bar{x}}^x |u^m(t)\,dt| \le a|x - \bar{x}| \le M|x - \bar{x}|$.

Thus, we have $T : S(\rho) \to S(\rho)$.
Now, for all $u, v \in S(\rho)$ we have

$$\left|(Tu)(x)-(Tv)(x)\right| \le \int_0^x \left|u^m(t)-v^m(t)\right|dt$$
$$\le aT_m \left|u-v\right|_S < \left|u-v\right|_S.$$

Hence, by the Banach Contraction Principle, T has a unique fixed point.

The above theorem shows that there exists a unique solution to the problems (3)-(4). However, it does not tell us how to find this solution. To find the power series solution of the problems (3)-(4), we will define the following approximating sequence

$$y_{n+1}(x) = c + \int_0^x y_n^m(t)dt \qquad (6)$$

where n = 0, 1, 2, ... and $y_0(x)$ is fixed functions of the class c^1 mapping from [0.a] to [0, a] such that $\left|y_0'(x)\right| \le a$. Then we have the following theorem.

Theorem 2 If the assumptions of the theorem 1 are satisfied then the sequences defined in (6) converges uniformly to the (unique) solution of the problems (3)-(4).

Proof We put $Y_k = \max_{x\varepsilon[0,a]}\left|y_k(x)-y_{k-1}(x)\right|$.

Then $Y_1 = \max_{x\varepsilon[0,a]}\left|y_1(x)-y_0(x)\right|$

$$= \max_{x\varepsilon[0,a]}\left|c+\int_0^x y_0^m(t)dt - y_0(x)\right|.$$

Since $y_0(x)$ maps from [0, a] to [0, a] then, we have

$$Y_1 \le c + a^2 \le a$$

and $Y_2 = \max_{x\varepsilon[0,a]}\left|y_2(x)-y_1(x)\right|$

$$= \max_{x\varepsilon[0,a]}\left|c+\int_0^x y_1^m(t)dt - c + \int_0^x y_0^m(t)dt\right|$$

$$= \max_{x\varepsilon[0,a]}\left|\int_0^x (y_1^m(t)-y_0^m(t))dt\right|$$

$$\le \max_{x\in[0,a]}\int_0^x \left|y_1^m(t)-y_0^m(t)\right|dt \le aY_1 \le a^2$$

and $Y_3 = \max_{x\varepsilon[0,a]}\left|y_3(x)-y_2(x)\right|$

$$= \max_{x\varepsilon[0,a]}\left|c+\int_0^x y_2^m(t)dt - c + \int_0^x y_1^m(t)dt\right|$$

$$= \max_{x\varepsilon[0,a]}\left|\int_0^x (y_2^m(t)-y_1^m(t))dt\right|$$

$$\le \max_{x\in[0,a]}\int_0^x \left|y_2^m(t)-y_1^m(t)\right|dt$$

$$\le aY_2 \le a^3.$$

Thus, we have $Y_k \le a^k$. Since $c + a^2 \le a$ then $a < 1$ when $c > 0$. Hence Y_k tends to zero as k tends to infinity. Since the family $\{Y_k\}$ is the Arzela-Ascoli family thus for every subsequence $\{y_{k_j}\}$ of $\{Y_k\}$ there exists a subsequence $\{y_{l_j}\}$ uniformly convergent and the limit needs to be a solution of the problem (3)-(4). Thus the sequence $\{Y_k\}$ tends uniformly to the (unique) solution of the problem (3)-(4).

It is easy to see that if $c = 0$ then the solution of (3)-(4) is identically zero. Thus we have the following theorem.

Theorem 3 Given the problem

$$y'(x) = y^m(x),\ x \in [0,a], \qquad (7)$$

with the initial condition

$$y(x) = 0. \qquad (8)$$

The solution of the problems (6)-(7) is $y(x) = 0$.

Proof (omitted).

Power Series Solutions

Second Degree Problem

We want to find the solution of the problem

$$y'(x) = y^2(x),\ x \in [0, 0.5] \qquad (9)$$

with the initial condition

$$y(0) = 0.25 \qquad (10)$$

Hence, we have $m = 2$, $a = 0.5$ and $c = 0.25$.
Thus $c + a^2 = 0.25 + 0.25 = 0.5 = a$.
If we let $y_0(x) = 0.25$

then $y_1(x) = 0.25 + \int_0^x 0.25dt = 0.25 + 0.25x$

and $y_2(x) = 0.25 + \int_0^x (0.25 + 0.25(0.25 + 0.25t))dt$

$$= 0.25 + \int_0^x (0.3125 + 0.0625t)dt$$

$$= 0.25 + 0.3125x + 0.03125x^2$$

and

$$y_3(x) = 0.25 + \int_0^x (0.25 + 0.3125(0.25 + 0.3125t + 0.03125t^2)$$
$$+ 0.03125(0.25 + 0.3125t + 0.03125t^2)^2)dt$$
$$= 0.25 + (3.3008 \times 10^{-1})x + (5.1270 \times 10^{-2})x^2 + (4.4352 \times 10^{-3})x^3 + (1.5258 \times 10^{-4})x^4 + (6.1035 \times 10^{-5})x^5$$

$$y_4(x) = 0.25 + \int_0^x y_3(y_3(t))dt$$
$$= 0.25 + (3.3579 \times 10^{-1})x + (5.5944 \times 10^{-3})x^2 + (5.8757 \times 10^{-3})x^3 + (5.3309 \times 10^{-4})x^4 + (7.7315 \times 10^{-5})x^5 + (8.3894 \times 10^{-6})x^6 + (7.8031 \times 10^{-7})x^7 + (6.5693 \times 10^{-8})x^8 + (5.2159 \times 10^{-9})x^9 + (3.8982 \times 10^{-10})x^{10} + (2.8083 \times 10^{-11})x^{11} + (1.9448 \times 10^{-12})x^{12} + (1.2936 \times 10^{-13})x^{13} + (8.1680 \times 10^{-15})x^{14} + (4.8066 \times 10^{-16})x^{15} + (2.6037 \times 10^{-17})x^{16} + (1.2803 \times 10^{-18})x^{17} + (5.6616 \times 10^{-20})x^{18} + (2.2291 \times 10^{-21})x^{19} + (7.7222 \times 10^{-23})x^{20} + (2.3235 \times 10^{-24})x^{21} + (5.95222 \times 10^{-26})x^{22} + (1.2623 \times 10^{-27})x^{23} + (2.1290 \times 10^{-29})x^{24} + (2.5849 \times 10^{-31})x^{25} + (1.9884 \times 10^{-33})x^{26}.$$

Figure 1 contains the graphs of $y(x) = \frac{e^x}{4}$, the fourth iterate $y_4(x)$ and the line $y(x) = \frac{1}{4} + \frac{x}{2}$. The curve of $y(x) = \frac{e^x}{4}$ is the dashed curve, the curve $y_4(x)$ is the dotted curve and the curve of the line $y(x) = \frac{1}{4} + \frac{x}{2}$ is the solid line.

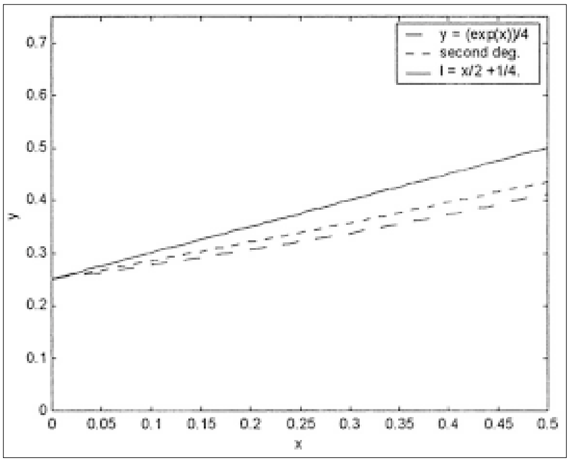

Fig 1.

Third Degree Problem

We want to find the solution of the problem

$$y'(x) = y^3(x), x \in [0, 0.5], \qquad (11)$$

with the initial condition

$$y(0) = 0.25. \qquad (12)$$

Hence, we have $m = 3$, $a = 0.5$ and $c = 0.25$.
Thus $c + a^2 = 0.25 + 0.25 = 0.5 = a$.
If we let $y_0(x) = 0.25$

then $y_1(x) = 0.25 + \int_0^x y_0^3(t)dt$
$= 0.25 + 0.25x$

and $y_2(x) = 0.25 + \int_0^x y_1^3(t)dt$
$= 0.25 + 0.328125x + 0.0078125x^2$

and $y_3(x) = 0.25 + \int_0^x y_2^3(t)dt$

$$= 2.5 \times 10^{-1} + 3.5997 \times 10^{-1}\, x + 1.8157 \times 10^{-2}\, x^2 + 4.1257 \times 10^{-4}\, x^3 + 4.7996 \times 10^{-6}\, x^4 + 6.3863 \times 10^{-8}\, x^5 + 4.9360 \times 10^{-10}\, x^6 + 5.4468 \times 10^{-12}\, x^7 + 3.7303 \times 10^{-14}\, x^8 + 1.97373 \times 10^{-16}\, x^9.$$

Figure 2 contains the graphs of $y(x) = \frac{e^x}{4}$, the third iterate $y_3(x)$ and the line $y(x) = \frac{1}{4} + \frac{x}{2}$. The curve of $y(x) = \frac{e^x}{4}$ is the dashed curve, the curve of $y_3(x)$ is in the dotted curve and the curve of the line $y(x) = \frac{1}{4} + \frac{x}{2}$ is the solid line.

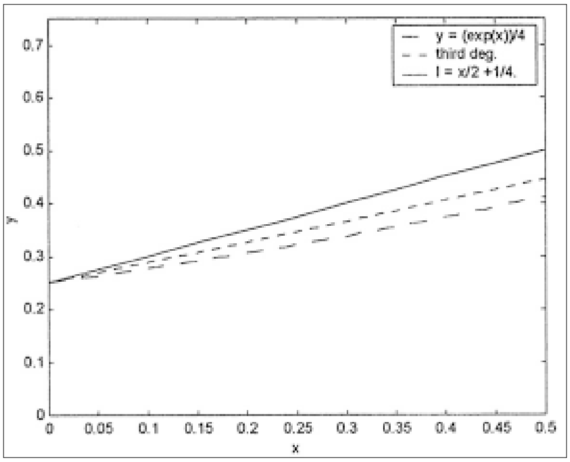

Fig 2.

Fourth Degree Problem

We want to find the solution of the problem

$$y'(x) = y^4(x),\ x \in [0, 0.5] \qquad (13)$$

with the initial condition

$$y(0) = 0.25. \qquad (14)$$

Hence, we have $m = 4$, $a = 0.5$ and $c = 0.5$

Thus $c + a^2 = 0.25 + 0.25 = 0.5 = a$.

If we let $y_0(x) = 0.25$

then $y_1(x) = 0.25 + \int_0^x y_0^4(t)\,dt$

$= 0.25 + 0.25x$

and $y_2(x) = 0.25 + \int_0^x y_1^4(t)\,dt$

$= 0.25 + 0.34765625x + 0.001953125x^2$

and $y_3(x) = 0.25 + \int_0^x y_2^4(t)\,dt$

$= 2.5 \times 10^{-1} + 3.7799 \times 10^{-1}\, x + 7.3828 \times 10^{-3}\, x^2$
$+ 4.1740 \times 10^{-5}\, x^3 + 1.3307 \times 10^{-7}\, x^4$
$+ 4.7601 \times 10^{-10}\, x^5 + 1.2181 \times 10^{-13}\, x^6$
$+ 3.9095 \times 10^{-16}\, x^7 + 7.2229 \times 10^{-18}\, x^8$
$+ 2.5991 \times 10^{-21}\, x^9 + 6.1761 \times 10^{-24}\, x^{10}$
$+ 1.5977 \times 10^{-26}\, x^{11} + 3.5761 \times 10^{-29}\, x^{12}$
$+ 7.2689 \times 10^{-32}\, x^{13} + 1.2649 \times 10^{-34}\, x^{14}$
$+ 1.9174 \times 10^{-37}\, x^{15} + 2.0433 \times 10^{-40}\, x^{16}$
$+ 1.3505 \times 10^{-44}\, x^{17}$.

Figure 3 contains the graphs of $y(x) = \frac{e^x}{4}$, the third iterate $y_3(x)$ and the line $y(x) = \frac{1}{4} + \frac{x}{2}$. The curve of $y(x) = \frac{e^x}{4}$ is the dashed curve, the curve of $y_3(x)$ is the dotted curve and the curve of the line $y(x) = \frac{1}{4} + \frac{x}{2}$ is the solid line.

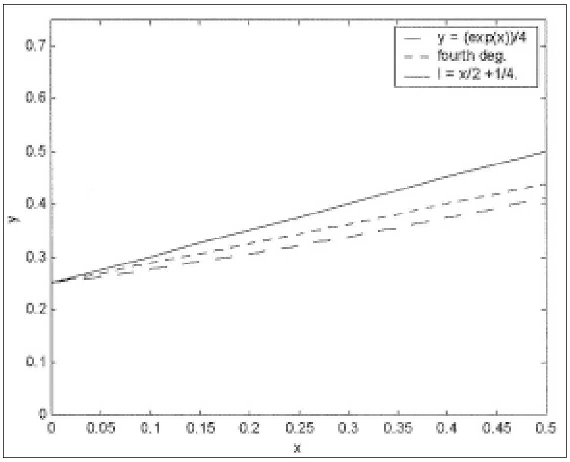

Fig 3.

We can see that the solution of the problem

$$y'(x) = y^m(x),\ x \in [0, 0.5], \qquad (15)$$

with the initial condition

$$y(0) = \frac{1}{4} \qquad (16)$$

is in the form of power series

$$y(x) = \frac{1}{4} + a_1 x + a_2 x \ + a_3 x^3 + \ldots + a_n x^n + \ldots\ . \qquad (17)$$

The following lemmas are needed for proving the existence and uniqueness of the global solutions.

Lemma 1 The relation of the a_k's in (17) is $a_1 > a_2 > a_3 > \ldots > a_n > \ldots$.

Proof It is easy to see, from (6), that $a_1 > a_2 > a_3 > \ldots > a_n > \ldots$.

Lemma 2 The coefficients a_n in (17) are less than or equal to $\frac{1}{2}$ for all n.

Proof Since $y(x) = \frac{1}{4} + a_1 x + a_2 x \ + a_3 x^3 + \ldots + a_n x^n + \ldots$

then $y'(x) = a_1 + 2a_2 x + \ldots$.

But $y'(x) = y^m(x)),\ x \in [0, 0.5]$ then $y'(x) \le \frac{1}{2}$ for all $x \in [0, 0.5]$ and $y'(0) \le \frac{1}{2}$, so $a_1 \le \frac{1}{2}$. Thus by lemma 1, we have $a_n \le \frac{1}{2}$ for all n.

Lemma 3 If $y^{(k)}(x)$ is the k^{th} derivative of $y(x)$, $y(x)$ being the power series of the problems (9)-(10), then $y^{(k)}(x) \le \frac{1}{2}$ for all k.

Proof From $y'(x) = y^2(x),\ x \in [0, 0.5]$, then

$y''(x) = y^3(x)y^2(x)$ thus $y''(x) \le \frac{1}{4}$.

Now $y'''(x) = y^4(x)y^3(x)y^2(x)y^2(x) + y^3(x)y^3(x)y^2(x)$

then $y'''(x) \le \frac{1}{2^3} + \frac{1}{2^4} \le \frac{1}{2}$.

It is easy to see that $y^{(k)}(x) \le \frac{1}{2^3} + \frac{1}{2^4} + ... + \frac{1}{2^i}$, where $i = 2 + (2 + 3 + 4 + ... + (k - 1))$. Thus we have $y^{(k)}(x) \le \frac{1}{2}$ for all k.

Lemma 4 If $y^{(k)}(x)$ is the k^{th} derivative of $y(x)$, $y(x)$ being the power series of the problems (11)-(12), then $y^{(k)}(x) \le \frac{1}{2}$ for all k.

Proof From $y'(x) = y^3(x), x \in [0, 0.5]$, then

$y''(x) = y^5(x)y^4(x)y^3(x)$. Then $y''(x) \le \frac{1}{2^3}$.

Now $y'''(x) = y^7(x)y^6(x)y^5(x)y^4(x)y^3(x) + y^5(x)y^6(x)y^5(x)y^4(x)y^3(x)y^3(x)$.

Then $y'''(x) \le \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7}$ then $y'''(x) \le \frac{1}{2}$.

It is easy to see that $y^{(k)}(x) \le \frac{1}{2^5} + \frac{1}{2^6} + ... + \frac{1}{2^i}$, where $i = 3 + (4 + 6 + 8 + ... + 2(k\text{-}1))$. Thus we have $y^{(k)}(x) \le \frac{1}{2}$ for all k.

Lemma 5 If $y^{(k)}(x)$ is the k^{th} derivative of $y(x)$, $y(x)$ is the derivative of, being the power series of the problem (13)-(14), then $y^{(k)}(x) \le \frac{1}{2}$ for all k.

Proof From $y'(x) = y^4(x), x \in [0, 0.5]$ then

$y''(x) = y^7(x)y^6(x)y^5(x)y^4(x)$ then $y''(x) \le \frac{1}{2^4}$.

Now $y'''(x) = y^{10}(x)y^9(x)y^8(x)y^7(x)y^6(x)y^5(x)y^4(x)y^6(x)y^5(x)y^4(x) + y^7(x)y^9(x)y^8(x)y^7(x)y^6(x)y^5(x)y^4(x)y^5(x)y^4(x) + y^7(x)y^6(x)y^8(x)y^7(x)y^6(x)y^5(x)y^4(x)y^4(x) + y^7(x)y^6(x)y^5(x)y^7(x)y^6(x)y^5(x)y^5(x)y^4(x)$

then $y'''(x) \le \frac{1}{2^7} + \frac{1}{2^8} + \frac{1}{2^9} + \frac{1}{2^{10}}$ then $y'''(x) \le \frac{1}{2}$.

It is easy to see that $y^{(k)}(x) \le \frac{1}{2^7} + \frac{1}{2^8} + ... + \frac{1}{2^i}$, where $i = 4 + (6 + 9 + 12 + ... + 3(k\text{-}1))$. Thus we have $y^{(k)}(x) \le \frac{1}{2}$ for all k.

Lemma 6 If $y^{(k)}(x)$ is the k^{th} derivative of $y(x)$, $y(x)$ being the power series of the problems (15)-(16), then $y^{(k)}(x) \le \frac{1}{2}$ for all k.

Proof From $y'(x) = y^m(x), x \in [0, 0.5]$, then

$y''(x) = y^{2m-1}(x)y^{2m-2}...y^{m+1}(x)y^m(x)$ thus $y''(x) \le \frac{1}{2^m}$.

Now $y'''(x) = y^{3m-2}(x)y^{3m-3}(x)\ldots y^m(x)y^{2m-2}(x)y^{2m-3}(x)\ldots y^m(x) + y^{2m-1}(x)y^{3m-3}(x)y^{3m-4}(x)\ldots y^m(x)y^{2m-3}(x)\ldots y^m(x) + y^{2m-1}(x)y^{2m-2}(x)y^{3m-4}(x)y^{3m-5}(x)\ldots y^m(x)y^{2m-4}(x)\ldots y^m(x) + ... + y^{2m-1}(x)y^{2m-2}(x)\ldots y^{m+1}(x)y^{2m-1}(x)y^{2m-2}(x)\ldots y^m(x)$.

Then $y'''(x) \le \frac{1}{2^{2m-1}} + \frac{1}{2^{2m}} + \frac{1}{2^{2m+1}} + ... + \frac{1}{2^{3m-2}}$ then $y'''(x) \le \frac{1}{2}$. It is easy to see that $y^{(k)}(x) \le \frac{1}{2^{2m-1}} + \frac{1}{2^{2m}} + ... + \frac{1}{2^i}$, where $i = m + (2(m - 1) + 3(m - 1) + (k - 1)(m - 1))$. Thus we have $y^{(k)}(x) \le \frac{1}{2}$ for all k.

Global Existence and Uniqueness Results

We will now study the global existence and uniqueness solutions of the problems of second, third, fourth and m^{th} degree equations.

Second Degree Problem.

The solution of the problems (9)-(10) is in the form of a power series $y(x) = a_0 + a_1x + a_2x^2 + a_3x^{3+} ... + a_nx^n + ...$ which converges for $x \in [0, 0.5]$. Then it converges for $x \in [-0.5, 0.5]$, so it has the Taylor's expansion and $a_n = \frac{y^{(n)}(0)}{n!}$.

But $y^{(n)}(x) \le \frac{1}{2}$ for all n (by lemma 3), so $y^{(n)}(0) \le \frac{1}{2}$ for all n. Thus we have $a_n = \frac{1}{2(n!)}$ for all n. Then $y(x) \le \frac{e^x}{2} - \frac{1}{4}$. Since $\frac{e^x}{2} - \frac{1}{4}$ has a Taylor's expansion that converges for $x \in (-\infty, \infty)$, then our power series solution converges for $x \in (-\infty, \infty)$. Thus we have the following theorem.

Theorem 4 There exists a unique solution to the problems (9)-(10) in the domain $[0, \infty)$.

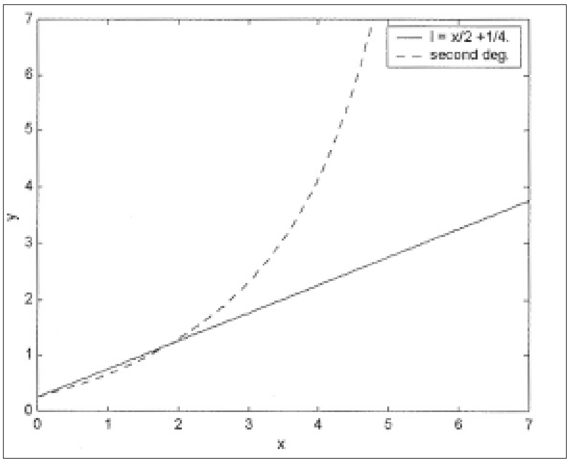

Fig 4.

Figure 4 contains the graph of the power series solution of the problems (9)-(10) using 10 terms in the series(the dashed curve) and the graph of the line $y(x) = \frac{1}{4} + \frac{x}{2}$ (the solid line).

Third Degree Problem

The solution of the problems (11)-(12) is of the form of power series $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$ converges for $x \in [0, 0.5]$. Then it converges for $x \in [-0.5, 0.5]$, so that it has a Taylor's expansion and $a_n = \frac{y^{(n)}(0)}{n!}$.

But $y^{(n)}(x) \le \frac{1}{2}$ for all n (by lemma 4), so $y^{(n)}(0) \le \frac{1}{2}$ for all n. Thus we have $a_n = \frac{1}{2(n!)}$ for all n. Then $y(x) \le \frac{e^x}{2} - \frac{1}{4}$. Since $\frac{e^x}{2} - \frac{1}{4}$ has a Taylor's expansion that converges for $x \in (-\infty, \infty)$, then our power series solution converges for $x \in (-\infty, \infty)$. Thus we have the following theorem.

Theorem 5 There exists a unique solution to the problems (11)-(12) in the domain $[0, \infty)$.

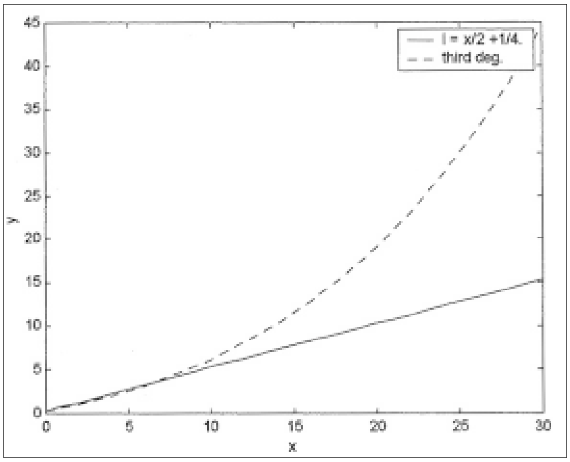

Fig 5.

Figure 5 contains the graph of the power series solution of the problems (11)-(12) using 10 terms in the series(the dashed curve) and the graph of the line $y(x) = \frac{1}{4} + \frac{x}{2}$ (the solid line).

Fourth Degree Problem

The solution of the problems (13)-(14) is of the form of a power series $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$ which converges for $x \in [0, 0.5]$. Then it converges for $x \in [-0.5, 0.5]$, so that it has a Taylor's expansion and $a_n = \frac{y^{(n)}(0)}{n!}$.

But $y^{(n)}(x) \le \frac{1}{2}$ for all n (by lemma 5), so $y^{(n)}(0) \le \frac{1}{2}$ for all n. Thus we have $a_n = \frac{1}{2(n!)}$ for all n. Then $y(x) \le \frac{e^x}{2} - \frac{1}{4}$. Since $\frac{e^x}{2} - \frac{1}{4}$ has a Taylor's expansion that converges for $x \in (-\infty, \infty)$, then our power series solution converges for $x \in (-\infty, \infty)$. Thus we have the following theorem.

Theorem 6 There exists a unique solution to the problems (13)-(14) in the domain $[0, \infty)$.

Figure 6 contains the graph of the power series solution of the problems (13)-(14) using 10 terms in the series(the dashed curve)and the curve of the line $y(x) = \frac{1}{4} + \frac{x}{2}$ (the solid line).

M^{th} Degree Problem

The solution of the problems (15)-(16) is of the

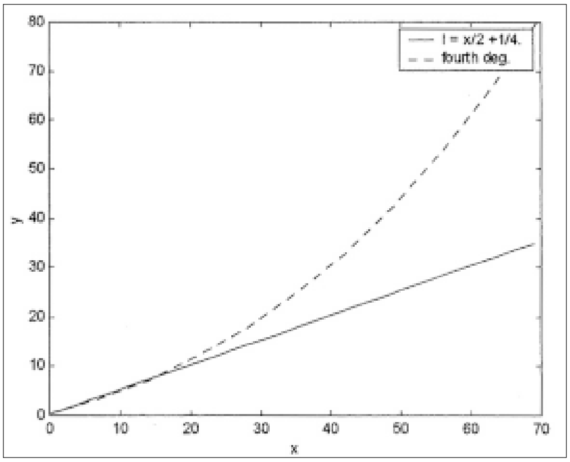

Fig 6.

form of a power series $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$ which converges for $x \in [0, 0.5]$. Then it converges for $x \in [-0.5, 0.5]$, so it has a Taylor's expansion and $a_n = \dfrac{y^{(n)}(0)}{n!}$.

But $y^{(n)}(x) \leq \dfrac{1}{2}$ for all n (by lemma 6), so $y^{(n)}(0) \leq \dfrac{1}{2}$ for all n. Thus we have $a_n = \dfrac{1}{2(n!)}$ for all n. Then $y(x) \leq \frac{e^x}{2} - \frac{1}{4}$. Since $\frac{e^x}{2} - \frac{1}{4}$ has a Taylor's expansion that converges for $x \in (-\infty, \infty)$, then our power series solution converges for $x \in (-\infty, \infty)$. Thus we have the following theorem.

Theorem 7 There exists a unique solution to the problems (15) - (16) in the domain $[0, \infty)$.

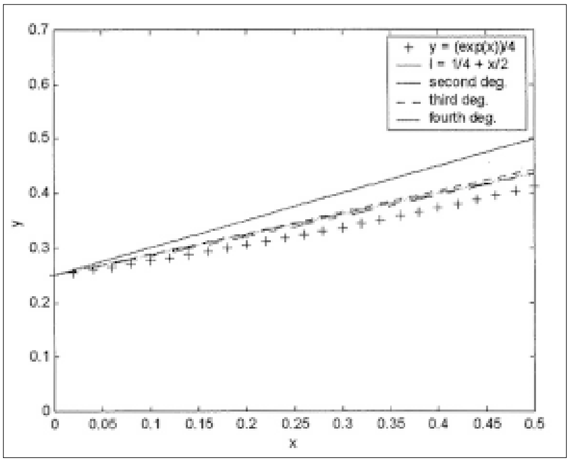

Fig 7.

Figure 7 contains the graph of the line $y(x) = \frac{1}{4} + \frac{x}{2}$ (the solid line), the graphs of the power series solution of the problems (9)-(10) (the dashdot curve), (11)-(12) (the dotted curve), (13)-(14) (the dash curve) and $y(x) = \dfrac{e^x}{4}$ (the plus line) using 10 terms in the series.

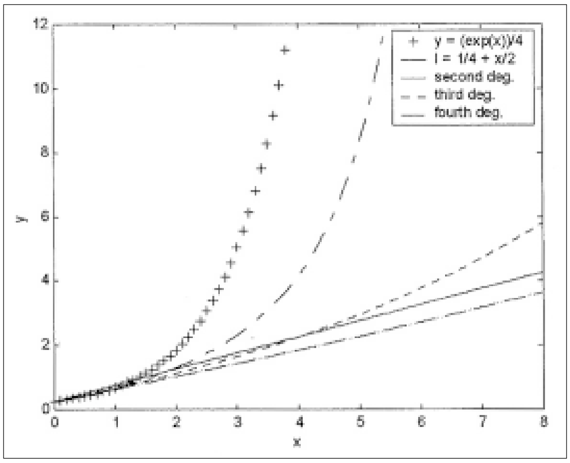

Fig 8.

Figure 8 contains the graph of the line (the solid line), the graphs of the power series solution of the problems (9)-(10) (the dashdot curve), (11)-(12) (the dotted curve), (13)-(14) (the dash curve) and the graph $y(x) = \dfrac{e^x}{4}$ (the plus curve) (T_2, T_3 and T_4 respectively) and using 10 terms in the series.

Conclusion

Letting E be the curve $y(x) = \dfrac{e^x}{4}$, T_k the graph of the solution of the k^{th} degree problem, L the curve $y(x) = \dfrac{1}{4} + \dfrac{x}{2}$, $P_{e,l} = (x_{e,l}, y_{e,l})$ the point of intersection of E and L, $P_{i,l} = (x_{i,l}, y_{i,l})$ the point of intersection of T_i and L, $P_{e,i} = (x_{i,j}, y_{i,j})$ the point of intersection of and, the point of intersection of T_i and T_j where $i < j$, then, we have

$x_{e,j} < x_{2,l} < x_{3,l} < x_{4,l} < ... < x_{m,l} < ...$
$y_{e,l} < y_{2,l} < y_{3,l} < y_{4,l} < ... < y_{m,l} < ...$
$x_{e,l} > x_{e,2} > x_{e,3} > x_{e,4} > ... > x_{e,m}$
$y_{e,l} > y_{e,2} > y_{e,3} > y_{e,4} > ... > y_{e,m}$
$x_{i,j} > x_{i,k}$ and $y_{i,j} > y_{i,k}$ when $j < k$
$x_{i,j} > x_{k,j}$ and $y_{i,j} > y_{k,j}$ when $i < k$.

Figure 9 shows how the graphs of E, T_2, T_3, T_4, ..., T_m, ... and L are related globally.

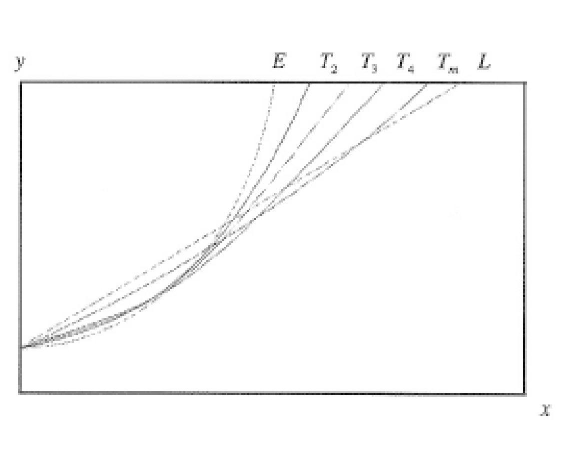

Fig 9.

There are two more things that need to be done. First, for each m we need to find the largest c so that the power series solution still converges uniformly. Second, for each *m* we need to find the exact power series solution of the problem.

The solution of the system

$$y'(x) = y^m(x),\ x \in [0, \infty), \qquad (20)$$

with the initial condition

$$y(0) = \frac{1}{4} \qquad (21)$$

as $m \to \infty$, is $y(x) = \frac{1}{4} + \frac{x}{2}$.

Finally, the solution $y = e^x$ of the problem $y'(x) = y(x),\ x \in [0, \infty)$, with the initial condition $y(0) = 1$, has been used in the modeling problems in mathematical applications. The suggestion here is that some of many areas of those applications may lead to the problem of the types (15)-(16) and its solution instead.

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