ScienceAsia 28 (2002) : **61-70**

The Inverse Problem for Euler's Equation on Two and Three Dimensional Lie Groups

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Received 18 Jun 2001

Accepted 26 Sep 2001

ABSTRACT Euler's equation, that describes geodesics for a left-invariant Riemannian metric on a Lie group associated with an inertia operator on its Lie algebra, plays an important role in mechanics. We examine the inverse problem of computing the inertia operator, up to multiplication by a constant, from a single solution of Euler's equation. We prove that, with exactly two exceptions, every two and three dimensional Lie group has the property that this inverse problem has a solution if and only if the angular velocity in the body does not lie in a proper subspace of the Lie algebra. The two exceptions are the group of Euclidean transformations of the plane and the product of the two-dimensional affine group with the group of real numbers.

KEYWORDS: geodesic, Lie algebra, Riemannian metric, co-adjoint representation, resultant.

INTRODUCTION

Euler[1] showed that the *inertial motion* of a rigid body about its center of mass is described by a function $g : \mathbb{R} \to SO(3)$ (Lie group of rotations), where the associated angular velocity in the body $\Omega : \mathbb{R} \to \mathbb{R}^3$ satisfies the equation

$$A\dot{\Omega} = A\Omega \times \Omega. \tag{1.1}$$

Here $A : \mathbb{R}^3 \to \mathbb{R}^3$ is the inertia operator (defined from the mass distribution of the body), Ω is defined by $\Omega \times \upsilon = g^{-1}\,\dot{g}\upsilon$, $\upsilon \in \mathbb{R}^3$, $\dot{\Omega}$, $\dot{g}$ is the derivative of Ω, g. We observe that $\mathbb{R}^3$, equipped with the vector cross product, is the Lie algebra of $SO(3)$:

Euler[2] also showed that the *inertial motion* of an incompressible inviscid fluid in a domain $D \subset \mathbb{R}^n$, $n \geq 2$ is described by a function $g : \mathbb{R} \to \mathrm{SDiff}(D)$ (infinite dimensional Lie group of volume preserving diffeomorphisms of D), where the associated velocity $u : D \to \mathbb{R}^3$, defined by the composition $u := \dot{g} \circ g^{-1}$, satisfies the equation

$$\dot{\Omega} + \nabla_u u = -\,grad\,p; \quad div\,u = 0; \quad u \cdot n = 0. \tag{1.2}$$

Here $p : D \to \mathbb{R}$ is a pressure function, $n : \partial D \to \mathbb{R}^3$ is the outward normal vector to D, and $\nabla_u u := u \cdot \nabla u$.

Moreau[3] observed that equations (1.1), (1.2) describe geodesics on $SO(3)$, $\mathrm{SDiff}(D)$ with respect to the left, right invariant Riemannian metric determined by an inertia operator on the associated Lie algebra. Arnold[4] developed the generalized Euler's equation

$$A\dot{u} = \sigma\, ad_u^*\, A u, \tag{1.3}$$

that describes geodesics on Lie groups with one-sided invariant Riemannian metrics.

Euler's equation plays an important role in mechanics because, according to the least action principle, it describes inertial motions of any dynamical system whose configuration manifold is a Lie group and whose kinetic energy is left, right invariant.[5,6] Arnold[7] used it to relate the sensitivity property of fluid flow to Riemannian curvature. Ebin, Marsden and Shkoller used it to derive existence, uniqueness, and regularity results for both Euler's and Navier-Stoke's equations.[8-10] The infinite dimensional group of diffeomorphisms of the circle plays an important role in loop groups and string theories.[11] Euler's equation for the standard right-invariant L^2 metric on its Lie algebra is Burger's equation, while Euler's equation for its one-dimensional Virasoro group extension with various metrics describes the Korteweg-de Vries and shallow water wave equations.[12,13] Fairlie, Fletcher and Zachos[14,15] and Zeitlin[16] used a sequence of geodesic flows on the special unitary groups $SU(n, C)$ to approximate ideal fluid flow on $\mathbb{T}^2$. The author[17] used these approximations to discuss wavelet bases for two-dimensional Euler flow. Bromberg[18] and Hermann[19] studied the existence of solutions of Euler's equation for indefinite metrics such as those that arise in general relativity.

Lawton and Noakes[20] addressed the inverse problem of computing the inertia operator, up to multiplication by a constant, from a single solution of Euler's equation and proved that for the Lie group $SO(3)$; a necessary and sufficient condition that this inverse problem have a solution is that the angular velocity in the body Ω be *nondegenerate* (not contained in a proper subspace of the Lie algebra $\mathbb{R}^3$). The importance of Euler's equation for general Lie groups motivated the work in this paper that addresses the inverse problem for Euler's equation on more general Lie groups.

Section 2 introduces basic concepts related to Euler's equation and derives two results valid for all Lie groups: first, if the inverse problem has a solution then the angular velocity in the body is nondegenerate; second, if both the angular velocity in the body and the angular velocity in space are nondegenerate, then the inverse problem has a solution.

Section 3 describes Milnor's classification of three dimensional Lie groups and uses it to prove the main result in this paper: the Lawton-Noakes result extends to all Lie groups having dimension two and three except for the unimodular group $E(2)$ (Euclidean transformations of the plane) and the nonunimodular group $A(1) \times \mathbb{R}$ (product of the two-dimensional affine group and the group of real numbers).

A preliminary account of this work has been presented in.[21]

EULER'S EQUATION

Lie Groups and their Algebras

We use Arnold's notation[4] (Appendix 2, Section A). G is a real Lie group, and for $g \in G$:

TG_g, T^*G_g is the tangent, cotangent space to G at g;

$TG := \cup_{g \in G} TG_g$, $T^*G := \cup_{g \in G} T^*G_g$ is the tangent, cotangent bundle of G;

$L_g, R_g : G \to G$, $g \in G$ is left, right multiplication by g;

$L_{g*}, R_{g*} : TG \to TG$ is the derivative of L_g, R_g;

$L^*_g, R^*_g : T^*G \to T^*G$ is the adjoint of L_{g*}, R_{g*}.

The Lie algebra $\mathfrak{g}$ is the tangent space to G at the identity I.

$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket product.

$\mathfrak{g}^*$ is the linear dual of $\mathfrak{g}$ consisting of all linear real-valued functions on $\mathfrak{g}^*$. If G is an infinite dimensional Lie group we will assume that these functions are continuous with respect to the specified topology on $\mathfrak{g}$.

$<\cdot, \cdot> : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the natural pairing.

$Lin(\mathfrak{g})$, $Lin(\mathfrak{g}^*)$ is the Lie algebra consisting of all linear mappings of $\mathfrak{g}, \mathfrak{g}^*$ into itself whose Lie bracket product is the commutator $[X, Y] := XY - YX$.

$GL(\mathfrak{g})$, $GL(\mathfrak{g}^*)$ is the group of all linear isomorphisms of $\mathfrak{g}, \mathfrak{g}^*$ onto itself.

Definition 2.1 *The adjoint, co-adjoint representation of G is the group homomorphism, antihomomorphism* $Ad : G \to GL(\mathfrak{g})$, $Ad^* : G \to GL(\mathfrak{g}^*)$ *described by*

$$Ad_g\, \upsilon := L_{g*} R_{g^{-1}*}\, \upsilon, \quad < Ad^*_g \ell, \upsilon > := < \ell, Ad_g\, \upsilon >,$$
$$g \in G,\ \ell \in \mathfrak{g}^*,\ \upsilon \in \mathfrak{g}.$$

The adjoint, coadjoint representation of $\mathfrak{g}$ *is the Lie algebra homomorphism, antihomomorphism* $ad : \mathfrak{g} \to \mathrm{Lin}(\mathfrak{g})$; $\mathrm{ad}^* : \mathfrak{g} \to Lin(\mathfrak{g}^*)$ *described by*

$$ad_u\, \upsilon := [u, \upsilon], \quad < ad^*_u \ell, \upsilon > := < \ell, ad_u\, \upsilon >,$$
$$\ell \in \mathfrak{g}^*,\ u, \upsilon \in \mathfrak{g}.$$

Lemma 2.1 *The coadjoint orbit* $Ad^*_G \ell := \{Ad^*_g \ell \mid g \in G\}$ *of each element* $\ell \in \mathfrak{g}^*$ *admits a nondegenerate symplectic structure.*

Proof This result is asserted in[4] (page 321) and proved in[4] (Appendix 5).

Corollary 2.1 *If* $dim(G) < \infty$ *then for every* $u \in \mathfrak{g}$, $dim(Ad_G\, u) < dim(G)$, *and for every* $\ell \in \mathfrak{g}^*$, *dim* $(Ad^*_G \ell)$ *is even.*

Proof For each $u \in \mathfrak{g}$ we construct the one parameter subgroup $H := \{e^{tu} : t \in \mathbb{R}\}$ and we observe that for every $g \in G$ and $h \in H$, $Ad_{gh}\, u = Ad_g\, u$. Therefore, the mapping χ defined by $\chi(gH) := Ad_g\, u$ is a differentiable mapping of the space G/H of left cosets of H in G, onto the orbit $Ad_G\, u$ of u under the adjoint representation of G. Therefore dim $(Ad_G\, u) \leq$ dim $(G/H) <$ dim(G) and the first assertion is proved.

The second assertion follows from lemma (2.1) and Darboux's theorem[4] (pages 230-232).

Definition 2.2 *A differentiable function* $g : \mathbb{R} \to G$ *(or trajectory in G) has associated velocity* $\omega := \dot{g} : \mathbb{R} \to TG$, *angular velocity in the body* $\omega_c := L_{g^{-1}*}\,\dot{g} : \mathbb{R} \to \mathcal{G}$, *and angular velocity in space* $\omega_s := R_{g^{-1}*}\,\dot{g} : \mathbb{R} \to \mathcal{G}$.

Lemma 2.2 $Ad_{e^{tu}} = e^{t\, ad_u}$ *and* $\frac{d}{dt} Ad_{e^{tu}} = Ad_{e^{tu}} \circ ad_u$, $u \in \mathcal{G}$.

Proof See[6] (Theorem 3.8, page 39).

Corollary 2.2 *If* $g : \mathbb{R} \to G$ *is differentiable then* $\frac{d}{dt} Ad_g = Ad_g \circ ad_{\omega_c} = ad_{\omega_s} \circ Ad_g$.

Proof This follows from lemma (2.2) and the approximation $g(t+\delta t) \approx g(t)e^{w_c \delta t}$.

Corollary 2.3 *If* $g : \mathbb{R} \to G$ *is once, twice, thrice differentiable then*

$$\omega_s = Ad_g \omega_c, \dot{\omega}_s = Ad_g \dot{\omega}_c, \ddot{\omega}_s = Ad_g([\omega_c, \dot{\omega}_c] + \ddot{\omega}_c).$$

Proof The first assertion follows immediately from definition (2.2). The second assertion follows from corollary (2.2) since

$$\dot{\omega}_s = \tfrac{d}{dt}(Ad_g \omega_c) = Ad_g[\omega_c, \omega_c] + Ad_g \dot{\omega}_c = Ad_g \dot{\omega}_c.$$

Similarly $\ddot{\omega}_s = \tfrac{d}{dt}(Ad_g \dot{\omega}_c) = Ad_g[\omega_c, \dot{\omega}_c] + Ad_g \ddot{\omega}_c$.

Definition 2.3 *Two trajectories* $g_1; g_2 : \mathbb{R} \to G$ *are equivalent if there exists* $h \in G$ *such that* $g_2 = hg_1$. *Two trajectories* $\omega_1, \omega_2, \mathbb{R} \to \mathcal{G}$ *are equivalent if there exists* $h \in G$ *such that* $\omega_2 = Ad_h\, \omega_1$.

Lemma 2.3 *Two trajectories in G have the same angular velocity in the body if and only if they are equivalent, and then their angular velocities in space are equivalent and either both angular velocities in space are degenerate or both are nondegenerate.*

Proof We assume that $g_1, g_2 : \mathbb{R} \to G$ have the same angular velocity in the body $\omega_c : \mathbb{R} \to \mathcal{G}$, we construct the vector field υ on the manifold $G \times \mathbb{R}$ by

$$\upsilon(g,t) = (L_{g*}\omega_c(t), \frac{\partial}{\partial t})\; g \in G, t \in \mathbb{R},$$

and we construct two trajectories $f_1(t) = (hg_1(t), t)$, where $h = g_2(0)g_1^{-1}(0)$, and $f_2(t) = (g_2(t), t)$. We observe that $f_1(0) = f_2(0)$, hence

$$\dot{f}_1(t) = (L_{h*}\dot{g}_1(t), \frac{\partial}{\partial t}) = (L_{h*}L_{g_1(t)*}\omega_c(t), \frac{\partial}{\partial t}) =$$

$$(L_{hg_1(t)*}\omega_c(t), \frac{\partial}{\partial t}) = \upsilon(f_1(t)) = \dot{f}_2(t).$$

Theorem 35.1 in[22] implies that there exists a one parameter subgroup of diffeomorphisms $D^t : G \times \mathbb{R} \to G \times \mathbb{R}$ G, $t \in \mathbb{R}$ for which υ is the phase velocity field. Therefore, $f_1(t) = D^t(f_1(0)) = D^t(f_2(0)) = f_2(t)$, $t \in \mathbb{R}$. This proves the first assertion. The second assertion follows from lemma (2.3).

Definition 2.4 *An inertia operator on* $\mathcal{G}$ *is a positive definite self-adjoint isomorphism* $A : \mathcal{G} \to \mathcal{G}^*$. *The scalar product* $(\cdot, \cdot) : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ *on* $\mathcal{G}$ *induced by A is defined by the* equation $(u, \upsilon) := \langle Au, \upsilon \rangle$, $u, \upsilon \in \mathcal{G}$.

Definition 2.5 *Let* $A : \mathcal{G} \to \mathcal{G}^*$ *be an inertia operator. Its associated left-invariant Riemannian metric* $A_g : TG_g \to T^*G_g$, $g \in G$ *is defined by the equation*

$$\langle A_g x, y \rangle := \langle AL_{g^{-1}*}x, L_{g^{-1}*}y \rangle, g \in G, x, y \in TG_g.$$

Definition 2.6 *A difierentiable function* $g : \mathbb{R} \to G$ *has associated (relative to the left-invariant Riemmanian metric associated to an inertia operator A on* $\mathcal{G}$*) momentum* $M_s := A_g\, \omega : \mathbb{R} \to T^*G$, *angular momentum in the body* $M_c := L_g^* M : \mathbb{R} \to \mathcal{G}^*$, *angular momentum in space* $M_s := R_g^* M : \mathbb{R} \to \mathcal{G}^*$, *and energy* $E := \frac{1}{2}\langle M_c, \omega_c \rangle$.

Lemma 2.4 *If G is a Lie group, A is an inertia operator on* $\mathcal{G}$, and $g : \mathbb{R} \to G$ *is differentiable, then* $M_c = A\omega_c$, $M_c = Ad_g^* M_s$, *and* $2E = \langle M_s, \omega_s \rangle$.

Proof These identities follow immediately from definitions (2.2) and (2.6).

Definition 2.7 *Let G be a Lie group and A be an inertia operator on $\mathcal{G}$. Euler's equation (associated to A) for a differentiable function $\phi : \mathbb{R} \to \mathcal{G}$ is*

$$A\dot{\phi} = ad^*_{\phi}\, A\phi. \qquad (2.1)$$

Lemma 2.5 *If G is a Lie group, A is an inertia operator on $\mathcal{G}$, $g : \mathbb{R} \to G$ is differentiable, and ω_c satisfies Euler's equation, then the energy E is constant.*

Proof $\dot{E} = \frac{d}{dt}\frac{1}{2} < A\omega_c, \omega_c > = < ad^*_{\omega_c}\, A\omega_c, \omega_c > =$

$< A\omega_c, [\omega_c, \omega_c] > = 0.$

Lemma 2.6 *If G is a Lie group, A is an inertia operator on $\mathcal{G}$, and $g : \mathbb{R} \to G$ is differentiable, then ω_c satisfies Euler's equation if and only if M_s is constant.*

Proof: This follows from corollary (2.2) and lemma (2.4) since for every $\upsilon \in \mathcal{G}$

$$< A\dot{\omega}_c, \upsilon > = \tfrac{d}{dt} < M_c, \upsilon > =$$

$$\tfrac{d}{dt} < Ad^*_g M_s, \upsilon > = \tfrac{d}{dt} < M_s, Ad_g \upsilon > =$$

$$< \dot{M}_s, Ad_g \upsilon > + < M_s, \tfrac{d}{dt} Ad_g \upsilon > =$$

$$< \dot{M}_s, Ad_g \upsilon > + < M_s, Ad_g \circ ad_{\omega_c} \upsilon > =$$

$$< \dot{M}_s, Ad_g \upsilon > + < Ad^*_g M_s, ad_{\omega_c} \upsilon > =$$

$$< \dot{M}_s, Ad_g \upsilon > + < ad^*_{\omega_c} A\omega_c, \upsilon >.$$

Corollary 2.4 *If G is a Lie group, A is an inertia operator on $\mathcal{G}$, $g : \mathbb{R} \to G$ is differentiable, and ω_c satisfies Euler's equation, then M_c is contained in the orbit of $M_s \in \mathcal{G}^*$ under Ad*.*

We describe a class of rank one self-adjoint operators on $\mathcal{G}^*$ and on $\mathcal{G}$. A vector $u \in \mathcal{G}$ induces a mapping $u : \mathbb{R} \to \mathcal{G}$ that is defined by the equation $u(t) := tu$, $t \in \mathbb{R}$, therefore its adjoint $u^* : \mathcal{G}^* \to \mathbb{R}$ is an element in $\mathcal{G}^{**}$ that is described by the equation $u^*\,\upsilon := < \upsilon, u >$, $\upsilon \in \mathcal{G}^*$. The mapping $u \to u^*$ defines a linear injection of $\mathcal{G}$ into $\mathcal{G}^{**}$ and we will regard $\mathcal{G}$ as a subspace of $\mathcal{G}^{**}$ under this injection. If $u \in \mathcal{G}$ then we can regard the composition $uu^* := u \circ u^* : \mathcal{G}^* \to \mathcal{G}$ as an operator $uu^* : \mathcal{G}^* \to \mathcal{G}^{**}$ that satisfies $< uu^*\,\upsilon, \omega > \; = < \upsilon, u > < \omega, u >$, $\upsilon, \omega \in \mathcal{G}^*$ and therefore is a positive-semidefinite self-adjoint operator on $\mathcal{G}^*$. It has rank one if and only if $u \neq 0$. Furthermore, if $\upsilon \in \mathcal{G}^*$ then the restriction of the composition $\upsilon\upsilon^* := \upsilon \circ \upsilon^* : \mathcal{G}^{**} \to \mathcal{G}^*$ to the subspace $\mathcal{G}$ of $\mathcal{G}^{**}$ satisfies $< \upsilon\upsilon^*\, u, \omega > = < \omega, u > < \upsilon, \omega >$, $u, \omega \in \mathcal{G}$ and therefore is a positive-semidefinite self-adjoint operator on $\mathcal{G}$. It has rank one if and only if $\upsilon \neq 0$. The following result follows immediately from definition (2.4) and the preceding discussion.

Lemma 2.7 *If G is a Lie group with dim(G) ≥ 2, $A : \mathcal{G} \to \mathcal{G}^*$ is an inertia operator on its Lie algebra, and $\upsilon \in \mathcal{G}^*$ is nonzero, then $\tilde{A} := A + \upsilon\upsilon^* : \mathcal{G} \to \mathcal{G}^*$ is an inertia operator that is not a constant multiple of A.*

The following result shows that the only if part of the Lawton-Noakes result[20] holds for all Lie groups.

Theorem 2.1 *If G is a Lie group whose dimension is ≥ 2, A is an inertia operator on $\mathcal{G}$, $g : \mathbb{R} \to G$ is differentiable, ω_c satisfies Euler's equation, and ω_c is contained in a proper closed subspace of $\mathcal{G}$, then there exists an inertia operator $\tilde{A} : \mathcal{G} \to \mathcal{G}^*$ that is not a multiple of A and such that ω_c satisfies Euler's equation for the left-invariant Riemmanian metric associated to $\tilde{A}$.*

Proof: Assume that the range of ω_c is contained in a proper subspace of $\mathcal{G}$. Then there exists a nonzero $\upsilon \in \mathcal{G}^*$ such that $< \upsilon, \omega_c > = 0$, hence $< \upsilon, \dot{\omega}_c > = 0$ and the operator $\tilde{A} : \mathcal{G} \to \mathcal{G}^*$, defined by $\tilde{A} := A + \upsilon\upsilon^*$, satisfies $\tilde{A}\dot{\omega}_c = ad^*_{\omega_c} \tilde{A}\omega_c$ since $\upsilon\upsilon^* \dot{\omega}_c = \upsilon\upsilon^* \omega_c = 0$. Furthermore, lemma (2.7) implies that the operator $\tilde{A}$ is an inertia operator that is not a constant multiple of A.

Theorem 2.2 *If G is a Lie group, A is an inertia operator on $\mathcal{G}$, $g : \mathbb{R} \to G$ is differentiable, ω_c satisfies Euler's equation, and both ω_c and ω_s are nondegenerate, then A can be computed, up to multiplication by a constant, from the values of ω_c over any interval* [a, b] $\subset \mathbb{R}$.

Proof It follows from lemma (2.3) that we may, without loss of generality, compute the trajectory g from ω_c as the unique solution of the differential equation $\dot{g} = L_{g*}\omega_c$ with initial condition $g(a) = I$, and compute the angular velocity in space by $\omega_s = \mathrm{Ad}_g\omega_c$. Then we compute the operators $\Phi, \Psi : \mathcal{G}^* \to \mathcal{G}$ by $\Phi := \int_a^b \omega_c(t)\omega_c^*(t)dt$ and $\Psi = \int_a^b \omega_s(t)\omega_s^*(t)dt$ and observe that they are invertible since both ω_c and ω_s are nondegenerate. Therefore, since the angular momentum in space satisfies the equation $M_s = Ad^*_{g^{-1}} A\omega_c$, we compute M_s (up to multiplication by a constant) from the equation

$$M_s \Psi = M_s \int_a^b \omega_s(t)\omega_s(t)^* \, dt =$$
$$\int_a^b < M_s, \omega_s(t) > \omega_s(t)^* \, dt = 2E \int_a^b \omega_s(t)^* \, dt.$$

and substitute it into the equation

$$A\Phi = A \int_a^b \omega_c(t)\omega_c(t)^* \, dt =$$
$$\int_a^b M_c(t)\omega_c(t)^* \, dt = \int_a^b (Ad_g^* M_s)\omega_c(t)^* \, dt$$

to compute A (up to multiplication by a constant).

Corollary 2.5 *If G is two dimensional and if ω_c is nondegenerate, then A can be computed, up to a constant multiple, from the values of ω_c over any interval $[a, b] \subset \mathbb{R}$.*

Proof Assume that ω_c is nondegenerate. Theorem (2.2) implies that it suffices to show that ω_s is nondegenerate. Assume to the contrary that ω_s is contained in a proper subspace V of $\mathcal{G}$. Then $\omega_c \in V \cap \{x \in \mathcal{G} \mid < M_s, x > = 2E\}$. Since $E > 0$ and $\mathcal{G}$ has dimension 2, this intersection has exactly one element. Therefore ω_s, and hence ω_c, is constant and this contradiction completes the proof.

The Inverse Problem for Three Dimensional Lie Groups

Throughout this section we make the following four assumptions:

1. G is a connected three dimensional Lie group with Lie algebra $\mathcal{G}$,
2. $A : \mathcal{G} \to \mathcal{G}^*$ is an inertia operator on $\mathcal{G}$,
3. $\omega_c : \mathbb{R} \to \mathcal{G}$ satisfies Euler's equation with respect to A,
4. ω_c is nondegenerate.

Our objective is to derive conditions on G, A, and ω_c that are both neccessary and sufficient for A to be determined, up to multiplication by a constant, from ω_c. We let $(\cdot, \cdot)$ denote the scalar product induced by A. We specify an orientation on $\mathcal{G}$ and let x denote the vector cross product on G with respect to $(\cdot, \cdot)$ and the specified orientation. Clearly, $(u, \upsilon \times \omega) = (u \times \upsilon, \omega)$, $u, \upsilon, \omega \in \mathcal{G}$. We use the following convention on indices: $a_i := a_j$ where $j = 1 + (i - 1) \bmod 3$. Thus $a_4 = a_1$. We choose a basis $\{e_1, e_2, e_3\}$ for $\mathcal{G}$ that is orthonormal with respect to $(\cdot, \cdot)$ and satisfies $e_i \times e_{i+1} = e_{i+2}$, $i = 1, 2, 3$. We define the linear operator $L : \mathcal{G} \to \mathcal{G}$ by $Le_{i+2} := [e_i, e_{i+1}]$, $i = 1, 2, 3$. Then

$$[u, \upsilon] = L(u \times \upsilon), u, \upsilon \in \mathcal{G} \qquad (3.1)$$

hence the operator L is independent of the choice of basis. However, choosing a different orientation has the effect of changing L into $-L$. We let $L^* : \mathcal{G}^* \to \mathcal{G}$ denote the adjoint of L and we let y; $[L]$ represent ω_c, L with respect to the basis $\{e_1, e_2, e_3\}$.

Lemma 3.1 $[A^{-1} L^* A] = [L]^T$.

Proof $[A^{-1}L^* A]_{ij} = (A^{-1}L^* Ae_j, e_i) = < L^* Ae_j, e_i >$
$= (e_j, Le_i) = [L]^T_{ij}, i, j = 1, 2, 3.$

Corollary 3.1 *Euler's equation for y is*

$$\dot{y} = -y \times [L]^T y. \qquad (3.2)$$

Proof Eulers equation for ω_c, equation (3.1), and lemma (3.1) imply that

$$\dot{y}_i = (\dot{\omega}_c, e_i) = (A^{-1}L^* A\omega_c, \omega_c \times e_i) =$$
$$(-\omega_c \times A^{-1}L^* A\omega_c, e_i) = (-y \times [L]^T y)_i, i = 1, 2, 3.$$

In addition to the previous assumptions, we further assume that $B : \mathcal{G} \to \mathcal{G}$ is an inertia operator on A such that ω_c satisfies Euler's equation with respect to B (in addition to satisfying Euler's equation with respect to A) and such that $< B\omega_c, \omega_c > = 2E$. We construct the operator $C := A^{-1}B : \mathcal{G} \to \mathcal{G}$ and we let $[C]$ represent C with respect to $\{e_1, e_2, e_3\}$. We let I denote the 3×3 identity matrix.

Lemma 3.2 $[C]^T = [C]$, $y^T y = y^T [C] y = 2E$, *and either* $[C] = I$ *or* $[C] - I$ *is nonsingular. The inertia matrix A is determined, up to multiplication by a constant, by* ω_w *if and only if our assumptions on B imply that* $[C] = I$.

Proof The first assertion follows from $[C]_{ij} = (Ce_j, e_i) = \langle Be_j, e_i \rangle = \langle Be_i, e_j \rangle = [C]^T_{ij}$, the second follows from $y^T y = (\omega_c, \omega_c) = \langle A\omega_c, \omega_c \rangle = 2E$; and the third follows from $y^T[C]y = (C\omega_c, \omega_c) = \langle B\omega_c, \omega_c \rangle = 2E$. Therefore, $y^T([C] - I)y = 0$, and since ω_c and therefore y is nonsingular, either $[C] - I = 0$ or $[C] - I$ is nonsingular. The final assertion follows from the fact that A is determined, up to multiplication by a constant, from ω_c if and only if our assumptions on B imply that there exists $\mu \in \mathbb{R}$ such that $B = \mu A$. Then $2E = \langle B\omega_c, \omega_c \rangle = \mu \langle A\omega_c, \omega_c \rangle = \mu 2E$, hence $\mu = 1$ andthe proof is concluded.

Lemma 3.3 *Let L and B satisfy the assumptions above. Then*

$$[C]\dot{y} = -y \times [L]^T [C] y. \tag{3.3}$$

Proof This follows from the following derivation

$$\begin{aligned}
\left([C]\dot{y}\right)_i = \left(C\dot{\omega}_c, e_i\right) &= \langle B\dot{\omega}_c, e_i \rangle = \\
\langle B\omega_c, [\omega_c, e_i] \rangle &= \langle B\omega_c, L(\omega_c \times e_i) \rangle = \\
\langle L^* B\omega_c, \omega_c \times e_i \rangle &= (A^{-1} L^* B\omega_c, \omega_c \times e_i) = \\
(-\omega_c \times A^{-1} L^* AC\omega_c, e_i) &= (-y \times [L]^T [C] y)_i, \ i = 1, 2, 3.
\end{aligned}$$

Lemma 3.4 *If* $[C] \neq I$ *then there exists a function* $R : \mathbb{R}^2 \to \mathbb{R}$ *where* $R = R_0 + R_2 + R_4$, R_j *is a homogeneous polynomial of degree j,* $R_2 \neq 0$, *and the components* y_2 *and* y_3 *of y satisfy*

$$R(y_2, y_3) = 0. \tag{3.4}$$

Proof We define $c_{ij} := [C]_{ij}$, $i, j = 1, 2, 3$, and construct polynomial functions $P, Q : \mathbb{R}^3 \to \mathbb{R}$ and $p_i, q_i, \mathbb{R}^2 \to \mathbb{R}$, $i = 0, 1, 2$, by

$$\begin{aligned}
P(x) &:= x^T x - 2E = x_1^2 + x_2^2 + x_3^2 - 2E = \\
&p_0(x_2, x_3)x_1^2 + p_1(x_2, x_3)x_1 + p_2(x_2, x_3) - 2E, \\
Q(x) &:= x^T [C] x - 2E = \sum_{i,j=2}^{3} c_{ij} x_i x_j - 2E = \\
&q_0(x_2, x_3)x_1^2 + q_1(x_2, x_3)x_1 + q_2(x_2, x_3) - 2E.
\end{aligned}$$

Hence

$$\begin{aligned}
&p_0(x_2, x_3) = 1, \quad p_1(x_2, x_3) = 0, \quad p_2(x_2, x_3) = x_2^2 + x_3^2, \\
&q_0(x_2, x_3) = c_{11} x_2^2, \quad q_1(x_2, x_3) = 2c_{12}x_2 + 2c_{13}x_3, \\
&q_2(x_2, x_3) = c_{22}x_2^2 + 2c_{23}x_2x_3 + c_{33}x_3^2.
\end{aligned}$$

Lemma (3.2) implies that $P(y) = Q(y) = 0$, therefore (Proposition 10.1[23]) implies that the components y_2 and y_3 of y satisfy equation (3.4) where

$$R := \det \begin{bmatrix} p_0 & p_1 & p_2 - 2_E & 0 \\ 0 & p_0 & p_1 & p_2 - 2E \\ q_0 & q_1 & q_2 - 2_E & 0 \\ 0 & q_0 & q_1 & q_2 - 2E \end{bmatrix}$$

is the *resultant* of polynomials P and Q, considered as polynomials in the variable x_1 whose coefficients are functions of x_2 and x_3. We compute

$$\begin{aligned}
&R = (q_2 - c_{11}p_2 + (c_{11} - 1)2E)^2 + (p_2 - 2E)q_1^2, \\
&R_0 = (c_{11} - 1)^2 4E^2, \\
&R_2 = (c_{11} - 1)4E(q_2 - c_{11}p_2) - 2Eq_1^2, \quad (3.5) \\
&R_4 = (q_2 - c_{11}p_2)^2 - p_2 q_1^2,
\end{aligned}$$

and observe that if $R_2 = 0$; then equation (3:4) implies that

$$c_{11} - 1 = (c_{22} - 1)y_2^2 + 2c_{23}y_2y_3 + (c_{33} - 1)y_3^2 = c_{12}y_2 + c_{13}y_3 = 0.$$

Since y is nondegenerate, $c_{11} = c_{22} = c_{33} = 1$, $c_{23} = c_{12} = c_{13} = 0$, and $[C] = I$.

Unimodular Groups

Definition 3.1 *A Lie group G is unimodular if its left invariant Haar measure is also right invariant.*

Lemma 3.5 *A connected Lie group G is unimodular if and only if either of the following equivalent conditions are satisfied: (i) det* $Ad_g = 1$, $g \in G$, *(ii) trace* $ad_x = 0$, $x \in \mathcal{G}$.

Proof See[24] (page 366).

Theorem 3.1 *G is unimodular if and only if* $[L]^T = [L]$, *and in this case, since the eigenvalues of L are real, the orientation on* $\mathcal{G}$ *can be chosen such that L has* *at least as many positive eigenvalues as negative eigenvalues. Furthermore, the basis* $\{e_1, e_2, e_3\}$ *can be chosen such that*

$$[L]=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}. \quad (3.6)$$

where the signs of the eigenvalues λ_i, $i = 1, 2, 3$ of L correspond to the six unimodular three dimensional Lie groups shown in Table 1.

If G is not unimodular we can choose the basis $\{e_1, e_2, e_3\}$ *such that* $\alpha + \delta > 0$ and

$$[L]=\begin{bmatrix}0 & 0 & 0\\ 0 & -\gamma & \alpha\\ 0 & -\delta & \beta\end{bmatrix}. \quad (3.7)$$

Table 1. Unimodular Three Dimensional Lie Groups.

λ_1	λ_2	λ_3	Lie group	description
+	+	+	$SU(2)$ or SO(3)	compact, simple
+	+	-	$SL(2, \mathbb{R})$ or $O(1; 2)$	noncompact, simple
+	+	0	Euclidean group $E(2)$	solvable
+	-	0	Lorentz group$E(1, 1)$	solvable
+	0	0	Heisenberg group $H(1)$	nilpotent
0	0	0	$\mathbb{R}^3$	commutative

Proof See Milnor.[25]

Lemma 3.6 *If L and B satisfy the assumptions above and* $[L]^T = [L]$, *then either* $[C] = I$ *or* $[L]$ *is a linear combination of* $[C]$ *and I: Therefore, the basis* $\{e_1, e_2, e_3\}$ *can be chosen such that* $[L]$ *satisfies equation (3.6) and* $[C]$ *is a diagonal matrix.*

Proof If $[L]^T = [L]$ then equation (3.2) implies that $\frac{d}{dt} y^T[L]y = 2\left(\left[L\right]^T y\right)^T \dot{y} = 0$ so there exists a constant $c \in \mathbb{R}$ such that $y^T[L]y = 2Ec$. Assume that $[C] - I \neq 0$. Then lemma (3.2) implies that $[C] - I$ is nonsingular, hence there exists an eigenvalue $\alpha \in \mathbb{R}$ and a generalized eigenvector e such that $([L] - cI)e = \alpha([C] - I)e$. Therefore the matrix $M := [L] - cI - \alpha([C] - I)$ is singular. Since $y^TMy = 0$ and y is nondegenerate, it follows that $M = 0$ and therefore $[L] = (c - \alpha) I + \alpha [C]$.

Theorem 3.2 *If G is unimodular and if* $G \neq E(2)$ *(the group of Euclidean motions of* $\mathbb{R}^2$*), then A is determined, up to multiplication by a constant, from* ω_c.

Proof Assume that G is unimodular, that L and B satisfy the assumptions above, and that $[C] \neq I$. Then theorem (3.1) and lemma (3.6) imply that we can choose the basis $\{e_1, e_2, e_3\}$ such that

$$[L]=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}, \quad [C]=\begin{bmatrix}c_1 & 0 & 0\\ 0 & c_2 & 0\\ 0 & 0 & c_3\end{bmatrix}$$

where the signs of $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are described by one of the six entries in Table 1 and $c_1, c_2, c_3 \in \mathbb{R}$. It suffices to prove that $c_1 = c_2 \neq c_3$, $\lambda_1 = \lambda_2 > 0$, and $\lambda_3 = 0$. We observe that equations (3.2) and (3.3) imply that the components y_1, y_2, y_3 of y satisfy the following equation

$$\begin{bmatrix}[c_1(\lambda_2-\lambda_3)-(\lambda_2 c_2-\lambda_3 c_3)]y_2 y_3\\ [c_2(\lambda_3-\lambda_1)-(\lambda_3 c_3-\lambda_1 c_1)]y_2 y_3\\ [c_3(\lambda_1-\lambda_2)-(\lambda_1 c_1-\lambda_2 c_2)]y_2 y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Furthermore, since y is nondegenerate, each of the functions y_2y_3, y_3y_1, y_1y_2 assumes a nonzero value hence

$$\begin{bmatrix}0 & c_1-c_2 & c_3-c_1\\ c_1-c_2 & 0 & c_2-c_3\\ c_3-c_1 & c_2-c_3 & 0\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}=0. \quad (3.8)$$

If $\lambda_j = 0$, $j = 1, 2, 3$ then $G = \mathbb{R}^3$ and Euler's equation has only constant solutions. This is impossible since we assumed that ω_c is nondegenerate. Therefore equation (3.8) implies that $2(c_1 - c_2)(c_3 - c_1)(c_2 - c_3) = 0$. This fact and the fact that $[C]$ is not a constant multiple of I implies that exactly one of the following conditions hold: (i) $c_1 = c_2 \neq c_3$, (ii) $c_1 = c_3 \neq c_2$, or (iii) $c_2 = c_3 \neq c_1$. Condition (ii) is impossible since it implies that $\lambda_1 = \lambda_3 \neq 0$ and $\lambda_2 = 0$ which is inconsistent with the signs in Table 1. Condition (iii) is impossible since it implies that $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 \neq 0$ which is inconsistent with the signs in Table 1. Therefore, condition (i) must hold and it implies that $\lambda_1 = \lambda_2 > 0$ and $\lambda_3 = 0$ hence $G = E(2)$.

Nonunimodular Groups

Lemma 3.7 *Assume that G is nonunimodular and that A, B, C, R, L and ω_c satisfy the assumptions stated previously in this section. Choose a basis $\{e_1, e_2, e_3\}$ for $\mathcal{G}$ such that $[L]$ satisfies equation (3.7). Define*

$$D=\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}$$

and construct functions $z_2, z_3 : \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{bmatrix} z_1(s) \\ z_2(s)\end{bmatrix} = exp(sD)\begin{bmatrix} y_2(0) \\ y_3(0)\end{bmatrix}, \quad s \in \mathbb{R}. \quad (3.9)$$

Then

$$R(z_2(s); z_3(s)) = 0, \quad s \in \mathbb{R}. \quad (3.10)$$

Proof Equation (3.2) implies that the components y_1, y_2, y_3 of y satisfy the following differential equation

$$\begin{bmatrix} \dot{y}_2 \\ \dot{y}_3\end{bmatrix} = y_1 D \begin{bmatrix} y_2 \\ y_3\end{bmatrix}. \quad (3.11)$$

Therefore,

$$\begin{bmatrix} y_2(t) \\ y_3(t)\end{bmatrix} = \exp(f(t)D) \begin{bmatrix} y_2(0) \\ y_3(0)\end{bmatrix}, \quad (3.12)$$

where $f : \mathbb{R} \to \mathbb{R}$ satisfies the differential equation $\dot{f} = y_1$. Since y is nondegenerate, y_1 is not identically zero so the range of f contains a nonzero open interval. This implies that equation (3.10) is satisfied over that open interval. Since R, z_2, z_3 are analytic, $R(z_2(s), z_3(s))$ is an analytic function of s and therefore vanishes for all values of s.

Theorem 3.3 *If G is nonunimodular then A is determined, up to multiplication by a constant, from ω_c if and only if G is not isomorphic to the product of the two-dimensional affine group with $\mathbb{R}$.*

Proof Let $A, B, C, D, R, L, \omega_c$ satisfy the assumptions in lemma (3.7). It suffices to prove that $[C] = I$. Let λ_1 and λ_2 denote the eigenvalues of D ordered so that $\Re\,\lambda_1 \geq \Re\,\lambda_2$. Clearly $\Re\,\lambda_1 > 0$ since theorem (3.1) implies that $\lambda_1 + \lambda_2 = \alpha + \delta > 0$. Therefore it suffices to consider the three cases: $\Re\,\lambda_2 > 0$, $\Re\,\lambda_2 = 0$, and $\Re\,\lambda_1 < 0$.

Case 1 $\Re\,\lambda_2 > 0$. Clearly $\lim_{s\to-\infty} |z_j(s)| = 0$, $j = 2, 3$ therefore equation (3.10) implies that $R_0 = R(0, 0) = 0$ and lemma (3.4) implies that $c_{11} = 0$ and

$$R = (p_2 - q_2)^2 + p_2 q_1^2 - 2Eq_1^2. \quad (3.13)$$

Furthermore, $\lim_{s\to\infty} |z_j(s)| = \infty$, $j = 2, 3$ therefore there exists s_0 such that

$$p_2(z_2(s); z_3(s)) > 2E, \text{ whenever } s > s_0. \quad (3.14)$$

If $q_1 \neq 0$ then, since y is nondegenerate, there exists a $s_1 > s_0$ such that

$$q_1(z_2(s_1), z_3(s_1)) \neq 0. \quad (3.15)$$

Equations (3.10), (3.13) and (3.15) imply that $2E \geq p_2(z_2(s_1)$ which contradicts equation (3.14) and proves that $q_1 = 0$. Then $R_2 = 0$ and lemma (3.4) implies that $[C] = I$.

Case 2 $\Re\,\lambda_2 = 0$. Clearly λ_1 and λ_2 are real and distinct therefore we can choose the basis $\{e_1, e_2, e_3\}$ such that $\alpha = \beta = 0$ and $z_2(s) = y_2(0) \neq 0$, $s \in \mathbb{R}$. This equation defines a plane and y must lie in the circle formed by intersecting this plane with the sphere defined by the equation $P(y) = 0$. This circle is orthogonal to and centered about the e_2 axis, therefore there exists a > 0 such that $y_2^2 = a(y_1^2 + y_3^2)$. The argument used in the proof of lemma (3.6) shows that either $[C] = I$, or that the matrix diag $([a - 1 \; a])$ is a linear combination of $[C]$ and I. Therefore $[C]$ is a diagonal matrix and $q_1 = 0$ and

$$(c_{11} - 2)2E + (c_{22} - c_{11})y_2^2 + (c_{33} - c_{11})y_3^2 = 0. \quad (3.16)$$

Since $y_2 = y_2(0) \neq 0$ and y_3 is not constant, $c_{33} = c_{11}$, $[C] = \mathrm{diag}([c_{11}\; c_{22}\; c_{11}])$, and

$$(c_{11} - 1)2E + (c_{22} - c_{11})y_2(0)^2 = 0. \quad (3.17)$$

We combine equations (3.2), (3.3), (3.7) and the fact that $\alpha = \beta = 0$ to obtain

$$\begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{11} \end{bmatrix} \left(y \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\gamma & -\delta \\ 0 & 0 & 0 \end{bmatrix} y \right) =$$

$$y \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\gamma & -\delta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{11} \end{bmatrix} y,$$

hence $\gamma y_2 y_3(c_{22} - c_{11}) = \gamma y_2 y_1(c_{22} - c_{11}) = 0$ and either $c_{22} - c_{11}$ or $\gamma = 0$. If $c_{22} = c_{11}$, then equation (3.17) implies that $[C] = I$, contradicting our initial hypothesis. Therefore $\gamma = 0$ and G is the product of the two-dimensional affine group with $\mathbb{R}$. Conversely, if G is the product of the two-dimensional affine group with $\mathbb{R}$, then we can choose a basis $\{e_1, e_2, e_3\}$ such that $\alpha = \beta = \gamma = 0$ and $\delta > 0$ and $C := A^{-1} B$ has the representation $[C] = \mathrm{diag}([c_{11}\ c_{22}\ c_{33}])$ where $c_{11}, c_{22} > 0$ satisfy equation (3.17) and $c_{11} \neq c_{22}$. This shows that A is not determined, up to multiplication by a constant, from ω_c.

Case 3 $\Re\, \lambda_2 < 0$. We choose a 2×2 matrix F such that $D = F^{-1}\, \mathrm{diag}([\lambda_1 \quad \lambda_2])F$ and define variables $\tilde{x}_j, j = 2, 3$

$$\begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} := F^{-1} \begin{bmatrix} x2 \\ x3 \end{bmatrix},$$

and polynomials $\tilde{R}_j(\tilde{x}_2, \tilde{x}_3) := R_j(x_2, x_3),\ j = 0, 2, 4,$ $\tilde{R} := \tilde{R}_0 + \tilde{R}_2 + \tilde{R}_4$. Clearly $\tilde{R}_j$ is homogeneous of degree j, and the functions $\tilde{z}_j$, $j = 2, 3$ defined by

$$\begin{bmatrix} \tilde{z}_2(s) \\ \tilde{z}_3(s) \end{bmatrix} := F^{-1} \begin{bmatrix} z_2(s) \\ z_3(s) \end{bmatrix}$$

satisfy $\tilde{z}_2(s) = e^{s\lambda_1} \tilde{z}_2(0), \quad \tilde{z}_3(s) = e^{s\lambda_2} \tilde{z}_3(0), s \in \mathbb{R}$, where $\tilde{z}_j(0) \neq 0, j = 2,3$, and $\tilde{R}(\tilde{z}_2(s), \tilde{z}_3(s)) = 0$. The polynomial $\tilde{R}$ is a sum of nine monomials. The preceeding equations and the inequalities $\lambda_1 > 0 > \lambda_2$ and $\lambda_1 + \lambda_2 > 0$ imply that $\lambda_1 = -3\lambda_2$ and that all the coefficients of $\tilde{R}$ vanish except for the constant term $\tilde{R}_0 = R_0$ and the coefficient of $\tilde{x}_2 \tilde{x}_3^3$. Therefore, $\tilde{R}_2 = 0$, hence $R_2 = 0$ and lemma (3.4) implies that $[C] = I$. This completes the proof.

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