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The Central Limit Theorems for Sums of Powers of Function of Independent Random Variables

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ABSTRACT Let (X_{nk}), $k = 1, ..., k_n$; $n = 1, 2, ...$ be a double sequence of infinitesimal random variables which are rowwise independent. In this paper, we give necessary and sufficient conditions for the sequence of distribution functions of $S_n^{(r)} = \left(g\left(X_{n1}\right)\right)^r + \cdots + \left(g\left(X_{nk_n}\right)\right)^r - B_n(r)$ to weakly converge to a limiting distribution function F_r for each natural number r, and also for convergence of (F_r).

KEYWORDS: central limit theorem, infinitely divisible, Le′vy's formula.

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INTRODUCTION

Let (X_{nk}), $k = 1, ..., k_n$; $n = 1, 2, ...$ be a double sequence of infinitesimal random variables which are rowwise independent. Let $S_n = X_{n1} + \cdots + X_{nk_n} - A_n$, where A_n are constants and let G_n be the distribution functions of S_n. Necessary and sufficient conditions for (G_n) to converge to a distribution function G are known, and in particular it is well known that G is infinitely divisible.

In 1957, Shapiro[1] considered the limit distribution functions of the sums $\left|X_{n1}\right|^r + \cdots + \left|X_{nk_n}\right|^r - B_n(r)$, where $B_n(r)$ are suitably chosen constants and $r \in \mathbf{N}$.

In 1974-1988, Shapiro[2-4] and Termwuttipong[5] gave the conditions which guarantees that the distribution functions of the sums $\left|X_1\right|^r + \left|X_2\right|^r + \cdots + \left|X_n\right|^r$ converge to a limit for $r < 0$.

In 1998, Neammanee[6] gave the conditions for convergence of distribution functions of $\left|\ln X_1\right|^r + \left|\ln X_2\right|^r + \cdots + \left|\ln X_n\right|^r$ for $r < 0$.

MAIN OF OBJECTIVE

In this work, we consider the distribution functions of the sums

$$S_n^{(r)} = \left(g\left(X_{n1}\right)\right)^r + \cdots + \left(g\left(X_{nk_n}\right)\right)^r - B_n(r)$$

where $r \in \mathbf{N}$ and $g : \mathbf{R} \to \mathbf{R}$ satisfies the following properties:

(g-1) $g(0) = 0$,

(g-2) g is continuous, strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$,

(g-3) there exist positive constants δ and c such that $\left|\frac{g(x)}{x}\right| < c$ for all $x \in (-\delta, \delta)$,

(g-4) $g(-\infty) = g(\infty) = \infty$.

Since g satisfies (g-1) and (g-2), we can write

$$g(x) = \begin{cases} g_1(x) & \text{if} \quad x \ge 0; \\ g_2(x) & \text{if} \quad x < 0. \end{cases}$$

where g_1 : $g_1 : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ defined by $g_1(x) = g(x)$ and $g_2 : \mathbf{R}_0^- \to \mathbf{R}_0^+$ defined by $g_2(x) = g(x)$. Since g is continuous at 0 and $g(0) = 0$, we can assume the δ in (g-3) has properties $g_1(\delta) < 1$ and $g_2(-\delta) < 1$. The followings are examples of g,

1. $g(x) = c|x|^n$ for $c > 0$ and $n \in \mathbf{N}$

2. $g(x) = \begin{cases} x + \sin x & \text{if} \quad x \ge 0; \\ -x + \sin x & \text{if} \quad x < 0. \end{cases}$

So Shapiro's results are our special case.

From now on, for $r \in \mathbf{N}$, we let $F_n^{(r)}, F_{nk}^{(r)}, F_{nk}$ be the distribution functions of $S_n^{(r)}, \left(g(X_{nk})\right)^r$ and X_{nk} respectively and for infinitely divisible distribution function F_r, we let $M_r, N_r, \gamma_r, \sigma_r^2$ be M, N, γ, σ^2 in

Le′vy's formula of F_r, (Petrov[7], chapter II). The necessary and sufficient conditions for convergence of the sequence of distribution functions of $S_n^{(r)}$ and the sequence of distribution functions F_r are given in Theorem A and Theorem B which stated below.

Theorem A Assume that $G_n \xrightarrow{w} G$ as $n \to \infty$. Then for each $r \in \mathbf{N}$ and for suitably chosen constants $B_n(r)$, $F_n^{(r)} \xrightarrow{w} F_r$ as $n \to \infty$ if and only if

1. $$\lim_{\varepsilon\to 0^+}\limsup_{n\to\infty}\sum_{k=1}^{k_n}\Big\{\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})}(g(t))^{2r}\,dF_{nk}(t)+\int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0}(g(t))^{2r}\,dF_{nk}(t^-)$$
$$-\Big(\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})}(g(t))^{r}\,dF_{nk}(t)+\int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0}(g(t))^{r}\,dF_{nk}(t^-)\Big)^2\Big\}=\sigma_r^2<\infty$$

and

2. $$\lim_{\varepsilon\to 0^+}\liminf_{n\to\infty}\sum_{k=1}^{k_n}\Big\{\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})}(g(t))^{2r}\,dF_{nk}(t)+\int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0}(g(t))^{2r}\,dF_{nk}(t^-)$$
$$-\Big(\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})}(g(t))^{r}\,dF_{nk}(t)+\int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0}(g(t))^{r}\,dF_{nk}(t^-)\Big)^2\Big\}=\sigma_r^2<\infty.$$

Theorem B Let $G_n \xrightarrow{w} G$ and $F_n^{(r)} \xrightarrow{w} F_r$ as $n \to \infty$ for all $r \in \mathbf{N}$. Then $F_r \xrightarrow{w} H$ and $r \to \infty$ if and only if

1. $M(x) = 0$ for all $x < g_2^{-1}(1)$
2. $N(x) = 0$ for all $x > g_1^{-1}(1)$
3. $\lim_{r\to\infty} \sigma_r^2 = (\sigma^*)^2$

where the functions M, N are functions in Le′vy's formula of F and σ^* is the constant in Le′vy's formula of H. Moreover, we know that

4. if $\sigma^* = 0$, M is continuous at $g_2^{-1}(1)$ and N is continuous at $g_1^{-1}(1)$ then H is degenerate
5. if $\sigma^* \neq 0$, M is continuous at $g_2^{-1}(1)$ and N is continuous at $g_1^{-1}(1)$ then H is normal
6. if $\sigma^* = 0$, M is discontinuous at $g_2^{-1}(1)$ or N is discontinuous at $g_1^{-1}(1)$ then H $(x - m)$ is Poisson, for some constant m
7. if $\sigma^* \neq 0$, M is discontinuous at $g_2^{-1}(1)$ or N is discontinuous at $g_1^{-1}(1)$ then H is the distribution function of the sum of two independent random variables one of which is normal and the other is Poisson.

Proofs of Main Results

Before we prove the main results we need the following lemmas.

Lemma 1 Let $X \sim N(a, \sigma^2)$ and $Y \sim \mathrm{Poi}(\lambda)$. If X and Y are independent, then Le′vy's formula of the characteristic function of $X + Y$ is

$$\log\varphi_{X+Y}(t) = it(a+\frac{\lambda}{2})t-\frac{1}{2}\sigma^2t^2+\int_0^\infty(e^{itx}-1-\frac{itx}{1+x^2})dK(x),$$

where $K: \mathbf{R}^+ \to \mathbf{R}$ is defined by $K(x)=\begin{cases}-\lambda & \text{if } 0<x\le 1;\\ 0 & \text{if } x>1.\end{cases}$

Proof Let φ_X and φ_Y be the characteristic functions of X and Y, respectively. From Lukacs[8] p93, we have

$\log\varphi_X(t) = iat - \frac{1}{2}\sigma^2t^2$ and $\log\varphi_Y(t) = i\frac{\lambda}{2}t + \int_0^\infty(e^{itx}-1-\frac{itx}{1+x^2})dK(x)$.

Since X and Y are independent,

$$\log\varphi_{X+Y}(t) = \log\varphi_X(t)\varphi_Y(t)$$
$$= \log\varphi_X(t) + \log\varphi_Y(t)$$
$$= iat-\frac{1}{2}\sigma^2t^2+i\frac{\lambda}{2}t+\int_0^\infty(e^{itx}-1-\frac{itx}{1+x^2})dK(x)$$
$$= it(a+\frac{\lambda}{2})t-\frac{1}{2}\sigma^2t^2+\int_0^\infty(e^{itx}-1-\frac{itx}{1+x^2})dK(x). \quad \#$$

Lemma 2 If $G_n \xrightarrow{w} G$ as $n \to \infty$ then for every $r \in \mathbf{N}$

1. $\lim_{n\to\infty}\sum_{k=1}^{k_n} F_{nk}^{(r)}(x) = 0$ for all $x < 0$ and
2. $\lim_{n\to\infty}\sum_{k=1}^{k_n}(F_{nk}^{(r)}(x)-1) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}}))$ a.e. on $(0,\infty)$.

Furthermore, if $F_n^{(r)} \xrightarrow{w} F_r$ for every $r \in \mathbf{N}$ then for each $r \in \mathbf{N}$, we have

3. $M_r = 0$ on $(-\infty, 0)$ and
4. $N_r(x) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}}))$ a.e. on $(0,\infty)$,

where M and N are functions in Le′vy's formula of F.

Proof Note that

$$F_{nk}^{(r)}(x)=\begin{cases}0 & \text{if } x<0;\\ P(X_{nk}=0) & \text{if } x=0;\\ F_{nk}(g_1^{-1}(x^{\frac{1}{r}}))-F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-) & \text{if } x>0\end{cases} \quad ...(2.1)$$

and $$F_{nk}^{(r)}(x)=\begin{cases}0 & \text{if } x<0;\\ F_{nk}^{(1)}(x^{\frac{1}{r}}) & \text{if } x\ge 0.\end{cases} \quad ...(2.2)$$

So 1. follows from (2.1). To prove 2, let $r\in \boldsymbol{N}$. Since $G_n \xrightarrow{w} G$, by Theorem 8 of Petrov[7] p81-82 we know that

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}F_{nk}(x)=M(x) \text{ and } \lim_{n\to\infty}\sum_{k=1}^{k_n}(F_{nk}(x)-1)=N(x) \quad ...(2.3)$$

for all continuity points of M and N. From (2.1) and (2.3)

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}(F_{nk}^{(r)}(x)-1)$$

$$=\lim_{n\to\infty}\sum_{k=1}^{k_n}\{F_{nk}(g_1^{-1}(x^{\frac{1}{r}}))-1-F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-)\}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{k_n}\{F_{nk}(g_1^{-1}(x^{\frac{1}{r}}))-1\}-\lim_{n\to\infty}\sum_{k=1}^{k_n}\{F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-)\}$$

$$=N(g_1^{-1}(x^{\frac{1}{r}}))-M(g_2^{-1}(x^{\frac{1}{r}})^-) \text{ a.e. on } (0,\infty)$$

$$=N(g_1^{-1}(x^{\frac{1}{r}}))-M(g_2^{-1}(x^{\frac{1}{r}})) \text{ a.e. on } (0,\infty).$$

Now, we suppose that $F_n^{(r)} \xrightarrow{w} F_r$ for every $r\in \boldsymbol{N}$. By (1), (2) and Theorem 8 of Petrov[7] p81-82 we have (3) and (4). #

Lemma 3 Assume that $F_n^{(r)} \xrightarrow{w} F_r$ for every $r\in \boldsymbol{N}$. Then for every $r\in \boldsymbol{N}$,

1. $M_r(x)=0$ on $(-\infty,0)$ and
2. $N_r(x)=N_1(x^{\frac{1}{r}})$ a.e. on $(0,\infty)$.

Proof We use the same argument in proving 2 of Lemma 2 by using (2.2) instead of (2.1). #

Lemma 4 Assume that

1. for every $r\in \boldsymbol{N}$, $F_n^{(r)} \xrightarrow{w} F_r$ as $n\to\infty$ and
2. $F_r \xrightarrow{w} H$ as $r\to\infty$.

Then H is one of the following

1. a degenerate distribution function
2. a Poisson distribution function
3. a normal distribution function
4. the distribution function of the sum of two independent random variables one of which is normal and the other is Poisson.

Proof Let r be any natural number. Then, by Lemma 3 we have $M_r = 0$ on $(-\infty, 0)$ and $N_r(x)=N_1(x^{\frac{1}{r}})$ a.e. on $(\infty, 0)$. Since $F_r \xrightarrow{w} H$ as $r\to\infty$, by Theorem 3 of Petrov[7] p75, we have

$\lim_{r\to\infty} M_r(x)=M^*(x)$ for all continuity points x of M^*,

$\lim_{r\to\infty} N_r(x)=N^*(x)$ for all continuity points x of N^*,

$\lim_{r\to\infty}\gamma_r=\gamma^*$ and

$$\lim_{\varepsilon\to 0^+}\limsup_{r\to\infty}\left\{\int_{-\varepsilon}^{0}u^2\,dM_r(u)+\sigma_r^2+\int_0^{\varepsilon}u^2\,dN_r(u)\right\}$$

$$=\lim_{\varepsilon\to 0^+}\liminf_{r\to\infty}\left\{\int_{-\varepsilon}^{0}u^2\,dM_r(u)+\sigma_r^2+\int_0^{\varepsilon}u^2\,dN_r(u)\right\}=(\sigma^*)^2$$

where M^*, N^*, γ^* and σ^* are associated with H in Le′vy's formula. This shows that

$$M^*=0 \text{ and } N^*(x)=\lim_{r\to\infty}N_1(x^{\frac{1}{r}})=\begin{cases}N_1(1^+) & \text{if } x>1;\\ N_1(1^-) & \text{if } 0<x<1.\end{cases}$$

But

$$N^*(\infty)=0, \text{ so } N_1(1^+)=0. \text{ Thus } N^*(x)=\begin{cases}0 & \text{if } x>1;\\ N_1(1^-) & \text{if } 0<x<1.\end{cases}$$

Case 1. $\sigma^* = 0$ and $N^* = 0$. Then H is degenerate.
Case 2. $\sigma^* \neq 0$ and $N^* = 0$. Then H is normal.
Case 3. $\sigma^* = 0$ and N^* takes one jump.

If $\gamma^*=-\dfrac{N_1(1^-)}{2}$ then H is Poisson.

If $\gamma^* \neq -\frac{N_1(1^-)}{2}$, let $m = -\frac{2\gamma^* + N_1(1^-)}{2}$, we note that the characteristic function $\varphi_m^*(t)$ of $H(x-m)$ is $e^{imt}\varphi^*(t)$, where $\varphi^*$is the characteristic function of H. Hence

$$\begin{aligned}\log\varphi_m^*(t) &= \log e^{imt}\varphi^*(t)\\ &= imt + \log\varphi^*(t)\\ &= imt + i\gamma^* t + 0 + 0 + \int_{-\infty}^{0^-}(e^{itx} - 1 - \frac{itx}{1+x^2})dN^*(x)\\ &= i(-\frac{N_1(1^-)}{2})t + 0 + 0 + \int_{-\infty}^{0^-}(e^{itx} - 1 - \frac{itx}{1+x^2})dN^*(x).\end{aligned}$$

So $H(x - m)$ is Poisson.

Case 4. $\sigma^* \neq 0$ and N^* takes one jump.
By Lemma 1, H is the distribution function of the sum of two independent random variables one of which is a Poisson and the other is a normal . #

Lemma 5 Assume that $F_n^{(r)} \xrightarrow{w} F_r$ as $n \to \infty$ for every $r \in N$ and $G_n \xrightarrow{w} G$ as $n \to \infty$. If $F_r \xrightarrow{w} H$ as $r \to \infty$ then

1. $M^*(x) = 0$ on $(-\infty, 0)$,
2. $N^*(x) = \begin{cases} 0 & \text{if } x > 1 ; \\ N(g_1^{-1}(1^-)) - M(g_2^{-1}(1^-)) & \text{if } 0 < x < 1, \end{cases}$
   on $(0, \infty)$ and
3. $M(g_2^{-1}(1^+)) = N(g_1^{-1}(1^+)) = 0$,

where M and N are functions in Le′vy's formula of F and $M^*$and N^* are functions in Le′vy's formula of H.

Proof Use the same technique in finding N^* and M^* in Lemma 4 by using Lemma 2 instead of Lemma 3. #

Proof of Theorem A

Note that, for ε > 0 we have

$$\int_{|x|<\varepsilon} x^2 dF_{nk}^{(r)}(x) - (\int_{|x|<\varepsilon} x dF_{nk}^{(r)}(x))^2$$

$$= \int_0^{\varepsilon} x^2 d[\ F_{nk}(g_1^{-1}(x^{\frac{1}{r}})) - F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-)\ ]$$

$$-(\int_0^{\varepsilon} x d[\ F_{nk}(g_1^{-1}(x^{\frac{1}{r}})) - F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-)\ ])^2$$

$$= \int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g(t_1))^{2r} dF_{nk}(t_1) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} (g(t_2))^{2r} dF_{nk}(t_2^-)$$

$$-(\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g(t_1))^{r} dF_{nk}(t_1) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} (g(t_2))^{r} dF_{nk}(t_2^-)\ )^2$$

$$[t_1 = g_1^{-1}(x^{\frac{1}{r}}) \text{ and } t_2 = g_2^{-1}(x^{\frac{1}{r}})]$$

$$= \int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g(t))^{2r} dF_{nk}(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} (g(t))^{2r} dF_{nk}(t^-)$$

$$-(\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g(t))^{r} dF_{nk}(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} (g(t))^{r} dF_{nk}(t^-)\ )^2. \qquad ...(2.4)$$

To prove necessity, we suppose that $F_n^{(r)} \xrightarrow{w} F_r$ as $n \to \infty$. Then 1. and 2. follow from Theorem 8 of Petrov[7] p81-82 and (2.4).

For sufficiency, we define $M_r : (-\infty, 0) \to \mathbf{R}$ and $N_r : (0, \infty) \to \mathbf{R}$ by

$$M_r(x) = 0 \text{ and } N_r(x) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}})).$$

Clearly, M_r and N_r are nondecreasing and $M_r(-\infty) = 0$, $N_r(\infty) = 0$. By (1) and (2) of Lemma 2 we have

$$\lim_{n\to\infty}\sum_{k=1}^{k_n} F_{nk}^{(r)}(x) = M_r(x) \text{ and} \qquad ...(2.5)$$

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}(F_{nk}^{(r)}(x) - 1) = N_r(x) \qquad ...(2.6)$$

for all continuity points of M and N. By assumptions 1, 2 and (2.4) we have

$$\lim_{\varepsilon\to 0^+}\limsup_{n\to\infty}\sum_{k=1}^{k_n}\{\ \int_{|x|<\varepsilon} x^2 dF_{nk}^{(r)}(x) - (\int_{|x|<\varepsilon} x dF_{nk}^{(r)}(x))^2\ \}$$

$$= \lim_{\varepsilon\to 0^+}\liminf_{n\to\infty}\sum_{k=1}^{k_n}\{\ \int_{|x|<\varepsilon} x^2 dF_{nk}^{(r)}(x) - (\int_{|x|<\varepsilon} x dF_{nk}^{(r)}(x))^2\ \} = \sigma_r^2 < \infty. \qquad ...(2.7)$$

By (2.5)-(2.7) and Theorem 8 of Petrov[7] p81-82, $F_n^{(r)} \xrightarrow{w} F_r$ as n → ∞. #

Proof of Theorem B

For $r \geq 2$ and $0 < \varepsilon < \min\{(g_1(\delta))^r, (g_2(-\delta))^r\}$, we have $\max\{g_1^{-1}(\varepsilon^{\frac{1}{r}}), |\ g_2^{-1}(\varepsilon^{\frac{1}{r}})\ |\} \leq \delta$ and

$$0 \leq \int_{-\varepsilon}^{0^-} u^2 dM_r(u) + \int_{0^-}^{\varepsilon} u^2 dN_r(u)$$

$$= 0 + \int_{0^+}^{\varepsilon} u^2 d\left[\ N(g_1^{-1}(u^{\frac{1}{r}})) - M(g_2^{-1}(u^{\frac{1}{r}}))\ \right]$$

(by Lemma 2 (3) and(4))

$$= \int_{0^+}^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g_1(t_1))^{2r}\, dN(t_1) - \int_{0^-}^{g_2^{-1}(\varepsilon^{\frac{1}{r}})} (g_2(t_2))^{2r}\, dM(t_2)$$

$$[t_1 = g_1^{-1}(u^{\frac{1}{r}}) \text{ and } t_2 = g_2^{-1}(u^{\frac{1}{r}})]$$

$$= \int_{0^+}^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g_1(t))^{2r}\, dN(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0^-} (g_2(t))^{2r}\, dM(t)$$

$$\le \varepsilon\left(\int_{0^+}^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g_1(t))^{r}\, dN(t)\right) + \varepsilon\left(\int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0^-} (g_2(t))^{r}\, dM(t)\right)$$

$$\le \varepsilon\left\{\int_{0^+}^{\delta} (g_1(t))^{r}\, dN(t) + \int_{-\delta}^{0^-} (g_2(t))^{r}\, dM(t)\right\}$$

$$\le \varepsilon\left\{\int_{0^+}^{\delta} (g_1(t))^{2}\, dN(t) + \int_{-\delta}^{0^-} (g_2(t))^{2}\, dM(t)\right\}$$

$$= \varepsilon\left\{\int_{0^+}^{\delta} \left(\frac{g_1(t)}{t}\right)^{2} t^2\, dN(t) + \int_{-\delta}^{0^-} \left(\frac{g_2(t)}{t}\right)^{2} t^2\, dM(t)\right\}$$

$$\le c^2\varepsilon\left\{\int_{0^+}^{\delta} t^2\, dN(t) + \int_{-\delta}^{0^-} t^2\, dM(t)\right\}.$$

(by property (g-3))...(2.8)

Then $0 \le \lim_{\varepsilon\to 0^+} \limsup_{r\to\infty} \left\{\ \int_{-\varepsilon}^{0^-} u^2\, dM_r(u) + \int_{0^+}^{\varepsilon} u^2\, dN_r(u)\ \right\}$

$$\le \lim_{\varepsilon\to 0^+} \limsup_{r\to\infty} c^2\varepsilon\left\{\int_{0^+}^{\delta} t^2\, dN(t) + \int_{-\delta}^{0^-} t^2\, dM(t)\right\} = 0.$$

Hence

$$\lim_{\varepsilon\to 0^+} \limsup_{r\to\infty} \left\{\ \int_{-\varepsilon}^{0^-} u^2\, dM_r(u) + \int_{0^+}^{\varepsilon} u^2\, dN_r(u)\ \right\} = 0 \text{....(2.9)}$$

Similarly, we have

$$\lim_{\varepsilon\to 0^+} \liminf_{r\to\infty} \left\{\ \int_{-\varepsilon}^{0^-} u^2\, dM_r(u) + \int_{0^+}^{\varepsilon} u^2\, dN_r(u)\ \right\} = 0 \text{....(2.10)}$$

To prove necessity, we suppose that $F_r \xrightarrow{w} H$ as $r \to \infty$. Since $G_n \xrightarrow{w} G$, by Theorem 8 of Petrov[7] p81-82 we have $\lim_{n\to\infty} \sum_{k=1}^{k_n} F_{nk}(x) = M(x)$ and $\lim_{n\to\infty} \sum_{k=1}^{k_n} \left[F_{nk}(x) - 1\right] = N(x)$ for all continuity points of M and N. Then (1) and (2) follow from Lemma 5(3) and the fact that M and N are nondecreasing and $M(-\infty) = N(\infty) = 0$.

Now, we will show (3).

Since $F_r \xrightarrow{w} H$, by Theorem 3 of Petrov[7] p75 we have

$$\lim_{\varepsilon\to 0^+} \limsup_{r\to\infty} \left\{\ \int_{-\varepsilon}^{0} u^2\, dM_r(u) + \sigma_r^2 + \int_{0}^{\varepsilon} u^2\, dN_r(u)\ \right\}$$

$$= \lim_{\varepsilon\to 0^+} \liminf_{r\to\infty} \left\{\ \int_{-\varepsilon}^{0} u^2\, dM_r(u) + \sigma_r^2 + \int_{0}^{\varepsilon} u^2\, dN_r(u)\ \right\} = (\sigma^*)^2$$

...(2.11)

By (2.9) - (2.11), we see that

$\limsup_{r\to\infty} \sigma_r^2 = (\sigma^*)^2$ and $\liminf_{r\to\infty} \sigma_r^2 = (\sigma^*)^2$.

So $\lim_{r\to\infty} \sigma_r^2 = (\sigma^*)^2$.

To prove sufficiency, we assume that (1), (2) and (3) hold.

Since $G_n \xrightarrow{w} G$ and $F_n^{(r)} \xrightarrow{w} F_r$ as $n \to \infty$, by Lemma 2, $M_r = 0$ and

$$N_r(x) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}})) \text{ a.e. on } (0,\infty).$$

Let $N^*: \mathbf{R}^+ \to \mathbf{R}$ be defined by $N^*(x) = \lim_{r\to\infty} N_r(x)$ and $M^*: \mathbf{R}^- \to \mathbf{R}$ be defined by $M^*(x) = \lim_{r\to\infty} M_r(x)$.

Then $M^* = 0$ on $(-\infty, 0)$ and by assumptions (1) and (2) $N^*(x) = \begin{cases} 0 & \text{if } x > 1\ ; \\ N(g_1^{-1}(1^-)) - M(g_2^{-1}(1^-)) & \text{if } 0 < x < 1 \end{cases}$

on $(0, \infty)$.

That is $M^*(-\infty) = N^*(\infty) = 0$. From assumption (3) and (2.9) we have

$$\lim_{\varepsilon\to 0^+} \limsup_{r\to\infty} \left\{\ \int_{-\varepsilon}^{0} u^2\, dM_r(u) + \sigma_r^2 + \int_{0}^{\varepsilon} u^2\, dN_r(u)\ \right\} = \lim_{r\to\infty} \sigma_r^2 = (\sigma^*)^2.$$

Similarly, we can show that

$$\lim_{\varepsilon\to 0^+} \liminf_{r\to\infty} \left\{\ \int_{-\varepsilon}^{0} u^2\, dM_r(u) + \sigma_r^2 + \int_{0}^{\varepsilon} u^2\, dN_r(u)\ \right\} = (\sigma^*)^2.$$

By Theorem 3 of Petrov[7] p75, we have $\lim_{r\to\infty} F_r(x) = H(x)$, where H is the infinitely divisible distribution determined by M^*, N^*, γ^* and $(\sigma^*)^2$.

By the same argument of Lemma 4 we have (4)-(7).

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References

1. Shapiro JM (1957) Sums of Powers of Independent Random Variables. *The American Mathematical Society* 29, 515-22.
2. Shapiro JM (1975) Domain of attraction reciprocals of power of random variables. *Siam Journal Appl Math* 29, 734-9.

3. Shapiro JM (1977) On domians of normal attraction to stable distributions. *Houston J Math* 3, 539-42.
4. Shapiro JM (1988) Limit distributions for sums of reciprocals of independent random variables, *Houston J Math* 14, 281-90.
5. Termwuttipong I (1986) Limit Distributions for Sums of the Reciprocal of a Positive Power of Independent Random Variables. *Ph D thesis*, Chulalongkorn Univ.
6. Neammanee K (1998) Limit Distributions for Random Sums of the Reciprocals of Logarithms of Independent Continuous Random Variables. *J Sci Res Chula Univ* 23, No 2.
7. Petrov VV (1975) Sum of Independent Random Variables, Springer-Varlag, NewYork.
8. Lukacs E (1970) Characteristic Functions. Hafner Publishing Company, NewYork.