RESEARCH ARTICLE

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Some Properties of the Kaprekar Numbers and a Means of Generation

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ABSTRACT This note describes what is believed to be a novel and easily implemented method for generating integer Kaprekar numbers. The starting point for the method is the observation made here, that a necessary condition for an integer k to be a Kaprekar number is that k must be congruent to k^2 modulo-9. Moreover, it is also shown here that a Kaprekar number k is either a member of the residue class [0] or residue class [1] modulo-9. For an integer k congruent to k^2 modulo-9, further steps are then established to find any integer values: q, r and n, such that $k = q + r$, and $k^2 = q \times 10^n + r$. The method described here is implemented using the computer algebra software package: Mathcad. A list of the entire integer Kaprekar numbers lying between 1 and 10^6 is generated. In addition, some results relating to the properties of the Kaprekar numbers are also presented.

KEYWORDS: Kapreka numbers, modulo arithmetic, congruent.

INTRODUCTION

The n-Kaprekar numbers are named after DR Kaprekar, who first introduced them in 1980.[1] They are formally defined as follows: an n-Kaprekar number k 1 (n = 1,2, ...) satisfies the pair of equations[2]:

$$k = q + r \qquad (1)$$

$$k^2 = q \times 10^n + r, \qquad (2)$$

where k, q, and r are positive integers, such that for $k > 1$, we require $q > 0$ and $0 < r < 10^n$.

By convention, zero is not considered to be an n-Kaprekar number, whereas k = 1 (where q = 0, r = 1, and n is arbitrary) is considered to be an n-Kaprekar. In addition, numbers of the form 10^m (where m 1 and requiring r = 0) are not considered to be n-Kaprekar numbers.[2]

Some justification for considering Kaprekar numbers as objects worthy of interest can perhaps be found from their inclusion in the Penguin Dictionary of Curious and Interesting Numbers.[3]

Some examples of n-Kaprekar numbers are:

$9^2 = 81$	$9 = 8+1$
$55^2 = 3025$	$55 = 30+25$
$95121^2 = 9048004641$	$95121 = 90480+4641.$

In fact, 9, 55 and 95121 are 1-Kaprekar, 2-Kaprekar and 5-Kaprekar numbers, respectively.

In this note, a new scheme for generating all the n-Kaprekar numbers less than or equal to a specified value is obtained.

APPLICATION OF MODULO-9 ARITHMETIC

The starting point for the approach presented here is the observation that one condition for an integer k to be an n-Kaprekar number, is that k must be congruent to k^2 modulo-9. Moreover, if k is an n-Kaprekar number, then it is a member of either the residue class [0] or the residue class [1] modulo-9.

Theorem 1: k is n-Kaprekar $\Rightarrow k^2 \equiv k \pmod{10^n - 1}$.

Proof: For an n-Kaprekar integer k defined by equations (1) and (2):

$$\begin{aligned} k^2 &= q \times 10^n + r \\ &= q \times (10^n - 1 + 1) + r \\ &= q \times (10^n - 1) + q + r \Rightarrow \\ k^2 &\equiv (q + r) \pmod{10^n - 1} \Rightarrow \\ k^2 &\equiv k \pmod{10^n - 1}. \qquad (3) \end{aligned}$$

Corollary: It immediately follows from the above that k is n-Kaprekar $\Rightarrow k^2 \equiv k \pmod 9$.

Remark: From the definition of the square of an n-Kaprekar integer k given above, it is observed that with the exception of k = 1 (for which q = 0, n is arbitrary, and r = 1) that $k^2 > 10^n$.

Theorem 2: For $t \equiv k (\mathrm{mod}\ 9)$, then $t \in \{[0], [1]\}$ for n-Kaprekar k.

Proof: For integers: k, s and t, we can write:

$$k = 9 \times s + t, 0 \le t \le 8 \quad (4)$$

where

$$t \equiv k (\mathrm{mod}\ 9), \quad (5)$$

Hence:

$$k^2 = 9^2 \times s^2 + 2 \times 9 \times s \times t + t^2 \Rightarrow$$
$$k^2 \equiv t^2 (\mathrm{mod}\ 9) \quad (6)$$

But it follows from (3) - (6) that if k is an n-Kaprekar number, then:

$$t \equiv t^2 (\mathrm{mod}\ 9), \quad 0 \le t \le 8. \quad (7)$$

But (7) is only true for t = 0 or t = 1. Hence the set of residue classes modulo-9 when k is n-Kaprekar is {[0],[1]}.

Theorem 3: q is even for all the n-Kaprekar numbers; and r is odd for the odd n-Kaprekar numbers and even for the even n-Kaprekar numbers.

Proof:
Case 1: n-Kaprekar k odd.

If k is odd and greater than 1, then from

$$k = q + r$$

either q could be odd and r even, or q could be even and r odd.

If k is odd, it follows that k^2 is odd. But

$$k^2 = q \times 10^n + r$$

implies because $n \ge 1$, that $q \times 10^n$ must be even. Hence r must be odd, and q must be even.

In the case of k = 1, it is noted that r = 1 and q = 0.

Case 2: n-Kaprekar k even.

If k is even then it follows that either both q or r could be even, or both q and r could be odd. But if k is even, it follows that k^2 is even. Hence, by similar reasoning to that given above for Case 1, both r and q must be even.

Method for Generating n-Kaprekar Numbers

For an integer k satisfying:

$$t \equiv k (\mathrm{mod}\ 9)$$
$$t \equiv t^2 (\mathrm{mod}\ 9), \quad (8)$$

where $t \in \{[0],[1]\}$, equations (1) and (2) can then be used to solve for possible values of r in terms of k and n, denoted here by r(n). Eliminating q gives:

$$r(n) = (k^2 - k \times 10^n)/(1 - 10^n)$$
for
$$n = 1, \ldots, n_{max}. \quad (9)$$

Hence another condition for k to be an n-Kaprekar number is that the fractional part of r(n) in equation (9) must be exactly equal to zero; that is, only integer values for r(n) are acceptable.

The exponent n in equation (9) ranges from 1 to a maximum value: n_{max}. The value of n_{max} is given by re-arranging equation (2). For r > 0 and q > 0, and for the monotonically increasing logarithmic function, we have:

$$10^n = (k^2 - r)/q \Rightarrow$$
$$n = \log_{10}[(k^2 - r)/q] \Rightarrow$$
$$n < \log_{10}[k^2] = 2 \log_{10}[k] \Rightarrow$$
$$n_{max} = \mathrm{ceil}(2 \log_{10}[k]) \quad (10)$$

where ceil(x) is defined here to be the smallest integer greater than or equal to the argument x.

Implementation of the Method

For a given integer k, (8) represents the first step in establishing if k is an n-Kaprekar number. Subsequent steps are represented by equations (9) and (10), and finally, if necessary, by the defining equations (1) and (2).

This method is implemented here using the computer algebra software package Mathcad.[4] The Mathcad code is shown later. A list of all the n-

Kaprekar Numbers lying between 1 and 10^6, generated using the method described here, is shown in the table below. The corresponding values for the integers: q, r and n, as defined by equations (1) and (2), are also included in the table.

Table 1. n-Kaprekar Numbers $1 \le k < 10^6$.

k	q	r	n	k^2
1	0	1	1	1
9	8	1	1	81
45	20	25	2	2025
55	30	25	2	3025
99	98	1	2	9801
297	88	209	3	88209
703	494	209	3	494209
999	998	1	3	998001
2223	494	1729	4	4941729
2728	744	1984	4	7441984
4879	238	4641	5	23804641
4950	2450	2500	4	24502500
5050	2550	2500	4	25502500
5292	28	5264	6	28005264
7272	5288	1984	4	52881984
7777	6048	1729	4	60481729
9999	9998	1	4	99980001
17344	3008	14336	5	300814336
22222	4938	17284	5	493817284
38962	1518	37444	6	1518037444
77778	60494	17284	5	6049417284
82656	68320	14336	5	6832014336
95121	90480	4641	5	9048004641
99999	99998	1	5	9999800001
142857	20408	122449	6	20408122449
148149	21948	126201	6	21948126201
181819	33058	148761	6	33058148761
187110	35010	152100	6	35010152100
208495	43470	165025	6	43470165025
318682	101558	217124	6	101558217124
329967	108878	221089	6	108878221089
351352	123448	227904	6	123448227904
356643	127194	229449	6	127194229449
390313	152344	237969	6	152344237969
461539	213018	248521	6	213018248521
466830	217930	248900	6	217930248900
499500	249500	250000	6	249500250000
500500	250500	250000	6	250500250000
533170	284270	248900	6	284270248900
538461	289940	248521	6	289940248521
609687	371718	237969	6	371718237969
627615	39390	588225	7	393900588225
643357	413908	229449	6	413908229449
648648	420744	227904	6	420744227904
670033	448944	221089	6	448944221089
681318	464194	217124	6	464194217124
791505	626480	165025	6	626480165025
812890	660790	152100	6	660790152100
818181	669420	148761	6	669420148761
851851	725650	126201	6	725650126201
857143	734694	122449	6	734694122449
961038	923594	37444	6	923594037444
994708	989444	5264	6	989444005264
999999	999998	1	6	999998000001

Table 2. Residue Classes Modulo-9 for n-Kaprekar Numbers k: $1 \le k < 10^6$.

i	k	$t \equiv k$ mod(9)	Prime Factorization of k
1	1	$\in [1]$	$=1$
2	9	$\in [0]$	$=3^2$
3	45	$\in [0]$	$=3^2 \times 5$
4	55	$\in [1]$	$=5 \times 11$
5	99	$\in [0]$	$=3^2 \times 11$
6	297	$\in [0]$	$=3^3 \times 11$
7	703	$\in [1]$	$=19 \times 37$
8	999	$\in [0]$	$=3^3 \times 37$
9	2223	$\in [0]$	$=3^2 \times 13 \times 19$
10	2728	$\in [1]$	$=2^3 \times 11 \times 31$
11	4879	$\in [1]$	$=7 \times 17 \times 41$
12	4950	$\in [0]$	$=2 \times 3^2 \times 5^2 \times 11$
13	5050	$\in [1]$	$=2 \times 5^2 \times 101$
14	5292	$\in [0]$	$=2^2 \times 3^3 \times 7^2$
15	7272	$\in [0]$	$=2^3 \times 3^2 \times 101$
16	7777	$\in [1]$	$=7 \times 11 \times 101$
17	9999	$\in [0]$	$=3^2 \times 11 \times 101$
18	17344	$\in [1]$	$=2^6 \times 271$
19	22222	$\in [1]$	$=2 \times 41 \times 271$
20	38962	$\in [1]$	$=2 \times 7 \times 11^2 \times 23$
21	77778	$\in [0]$	$=2 \times 3^2 \times 29 \times 149$
22	82656	$\in [0]$	$=2^5 \times 3^2 \times 7 \times 41$
23	95121	$\in [0]$	$=3^3 \times 13 \times 271$
24	99999	$\in [0]$	$=3^2 \times 41 \times 271$
25	142857	$\in [0]$	$=3^3 \times 11 \times 13 \times 37$
26	148149	$\in [0]$	$=3^4 \times 31 \times 59$
27	181819	$\in [1]$	$=11 \times 16529$
28	187110	$\in [0]$	$=2 \times 3^5 \times 5 \times 7 \times 11$
29	208495	$\in [1]$	$=5 \times 7^2 \times 23 \times 37$
30	318682	$\in [1]$	$=2 \times 7 \times 13 \times 17 \times 103$
31	329967	$\in [0]$	$=3^3 \times 11^2 \times 101$
32	351352	$\in [1]$	$=2^3 \times 37 \times 1187$
33	356643	$\in [0]$	$=3^4 \times 7 \times 17 \times 37$
34	390313	$\in [1]$	$=7 \times 11 \times 37 \times 137$
35	461539	$\in [1]$	$=13^2 \times 2731$
36	466830	$\in [0]$	$=2 \times 3^3 \times 5 \times 7 \times 13 \times 19$
37	499500	$\in [0]$	$=2^2 \times 3^3 \times 5^3 \times 37$
38	500500	$\in [1]$	$=2^2 \times 5^3 \times 7 \times 11 \times 13$
39	533170	$\in [1]$	$=2 \times 5 \times 11 \times 37 \times 131$
40	538461	$\in [0]$	$=3^3 \times 7^2 \times 11 \times 37$
41	609687	$\in [0]$	$=3^5 \times 13 \times 193$
42	627615	$\in [0]$	$=3^3 \times 5 \times 4649$
43	643357	$\in [1]$	$=11^2 \times 13 \times 409$
44	648648	$\in [0]$	$=2^3 \times 3^4 \times 7 \times 11 \times 13$
45	670033	$\in [1]$	$=7 \times 13 \times 37 \times 199$
46	681318	$\in [0]$	$=2 \times 3^3 \times 11 \times 31 \times 37$
47	791505	$\in [0]$	$=3^3 \times 5 \times 11 \times 13 \times 41$
48	812890	$\in [1]$	$=2 \times 5 \times 13^3 \times 37$
49	818181	$\in [0]$	$=3^5 \times 7 \times 13 \times 37$
50	851851	$\in [1]$	$=7 \times 11 \times 13 \times 23 \times 37$
51	857143	$\in [1]$	$=7 \times 122449$
52	961038	$\in [0]$	$=2 \times 3^3 \times 13 \times 37^2$
53	994708	$\in [1]$	$=2^2 \times 11 \times 13 \times 37 \times 47$
54	999999	$\in [0]$	$=3^3 \times 7 \times 11 \times 13 \times 37$

Casting Out Nines

The sequence of n-Kaprekar numbers is composed of two types of integers corresponding to one of the two residue classes: [0] or [1] modulo 9. A familiar (and easily proven) rule for calculating residue classes in modulo-9 arithmetic, often called "casting out nines," can be applied to the n-Kaprekar numbers. For example:

$$55 \pmod 9 \equiv (5+5) \pmod 9 \equiv 1$$

$$95121 \pmod 9$$
$$\equiv (9+5+1+2+1) \pmod 9 \equiv 0$$

For an n-Kaprekar number k, where:

$$k = a_p\, a_{p-1\ldots}\, a_2 a_1, \qquad (11)$$

meaning

$$k = a_1 + a_2 \times 10^1 + a_3 \times 10^2 + \ldots a_p \times 10^{p-1} \qquad (12)$$

we have for k either

$$(a_1 + a_2 + \ldots a_p) \pmod 9 \equiv 0 \qquad (13)$$

or

$$(a_1 + a_2 + \ldots a_p) \pmod 9 \equiv 1 \qquad (14)$$

A complete list of the residue classes modulo-9, in sequence, corresponding to the 54 n-Kaprekar numbers lying between 1 and 10^6 given in Table 1, is shown below in Table 2. The prime factorizations of the n-Kaprekar numbers are also given.

Conclusions

The n-Kaprekar numbers k are congruent to their squares modulo-9. Moreover, the set of residue classes modulo-9 is {[0],[1]}; hence the n-Kaprekar sequence is shown to be composed of two distinct types, categorized by these two residue classes. Also, some definite statements can be made about the evenness of q, and the oddness or evenness of r, where k = q + r.

The n-Kaprekar numbers occurring within a specified range can be generated easily, and relatively quickly, using the approach developed here.

References

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Mathcad Code

The Mathcad implementation of the method developed here is shown below:

```
k(kupper) := | j←0
             | nupper←ceil(2·log(kupper))
             | for k∈ 1..kupper
             |   | mod9k←mod(k, 9)
             |   | mod9k2←mod{mod9k^2, 9}
             |   | if (mod9k=mod9k2)
             |   |   | flag←0
             |   |   | for n∈ 1.. nupper
             |   |   |   | r← (k·10^n - k^2)/(10^n - 1)
             |   |   |   | if (0<r)·{r<10^n}·(floor(r) = r)·(flag = 0)
             |   |   |   |   | q←k - r
             |   |   |   |   | if {k^2 = q·10^n + r}·(k = q + r)
             |   |   |   |   |   | j←j + 1
             |   |   |   |   |   | x_{j,0}←k
             |   |   |   |   |   | x_{j,1}←q
             |   |   |   |   |   | x_{j,2}←r
             |   |   |   |   |   | x_{j,3}←n
             |   |   |   |   |   | x_{j,4}←q·10^n + r
             |   |   |   |   |   | x_{j,5}←k^2
             |   |   |   |   |   | flag←1
             | x
```

The n-Kaprekar numbers lying between 1 and 10^6, as well as all the additional information shown in Table 1, were generated using $k(10^6)$.