ScienceAsia 26 (2000) : **49-55**

Structure of Discrete-Time H^∞ Controller

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Received 2 Jun 1999
Accepted 4 Feb 2000

Abstract This paper is concerned with the investigation of the closed-loop structure of an H^∞ control system. It is shown that H^∞ controller is represented, like Linear Quadratic Gaussian control, as a pseudo state feedback, that is, a state feedback interconnected with an observer. However, in the H^∞ control problem (in particular, discrete-time cases) the control structure is more complicated since we cannot choose the state feedback and the observer independently.

KEYWORDS: discrete-time H^∞ control, chain-scattering representation, hoMographic transformation.

Introduction

H^∞ control theory has received much attention over the last two decades (see Francis,[4] Doyle *et al*,[3] Stoorvogel,[10] Mirkin[9] and the references therein). Early results for the H^∞ control problem were derived for the continuous-time case. However, in practical applications controllers operate mainly in discrete-time. We can use a discrete-time controller to control a continuous-time system. There are many results in this direction.[1-2,12] An alternative approach is discretizing the system first and then using H^∞ control designed for discrete-time systems. This might be a simpler approach. Also, certain systems are in themselves inherently discrete, and certainly for these systems it is useful to have results available for H^∞ control problem.

In a recent paper,[8] we studied H^∞ control for discrete-time systems. We have obtained a necessary and sufficient condition under which an H^∞ norm bound can be achieved by an internally stabilizing output feedback controller. In this paper, we investigate the structure of H^∞ controller in details and show its intrinsic pseudo-state feedback structure. Also, we derive a set of necessary and sufficient conditions for the existence of strictly proper H^∞ controllers. This problem has been studied before in Stoorvogel,[11] Mirkin.[9] However, by using the chain-scattering approach, our derivation is much simpler and it clarifies the controller structure in a straightforward way.

This paper is organized as follows. In Section 2, some mathematical preliminaries are briefly reviewed. Section 3 contains the main results. There, we clarify the structure of discrete-time H^∞ controller and the necessary and sufficient conditions for the existence of strictly proper controller. In Section 4 we discuss stability properties of the discrete-time controller. Most parts of these details have already been reported by Stoorvogel[10] and Green *et al*,[15] for completeness, they are repeated here. A collection of simple examples are given in Section 5.

Notations :

$$J_{mr} := \begin{bmatrix} I_m & 0 \\ 0 & -I_r \end{bmatrix},$$

$$J = J_{mr},\ J' = J_{pq},\ J'' = J_{mq},$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(zI - A)^{-1}B + D.$$

$\sigma(A)$ is the maximum spectral radius of constant matrix A, $\boldsymbol{R}_{m\times r}$ is the set of real m x r matrices, $\boldsymbol{RL}^\infty_{m\times r}$ is the set of all m x r rational matrices without pole on the unit circle, $\boldsymbol{RH}^\infty_{m\times r}$ is the set of all m x r rational stable proper matrices. $\boldsymbol{BH}^\infty_{m\times r}$ is the subset of $\boldsymbol{RH}^\infty_{m\times r}$ whose norm is less than 1.

Preliminaries and Problem Formulation

Plant

We consider a Linear time-invariant discrete-time system described by

$$x_{k+1} = Ax_k + B_1w_k + B_2u_k, \quad (2.1a)$$
$$z_k = C_1x_k + D_{11}w_k + D_{12}u_k, \quad (2.1b)$$
$$y_k = C_2x_k + D_{21}w_k, \quad (2.1c)$$

where z is the controlled error ($\dim(z) = m$), y is the observation output ($\dim(y) = q$), w is the exogeneous input ($\dim(w) = r$), u is the control input ($\dim(u) = p$).

We make the usual assumptions that

(A1) (A, B_2) is stabilizable and (A, C_2) is detectable. This assumption is necessary in order that the H^∞ control problem is solvable. In this paper, we deal with the so-called *standard problem* in which the following assumptions hold :

(A2) rank $D_{21} = q$, rank $D_{12} = p$.

Standard H^∞ Control Problem

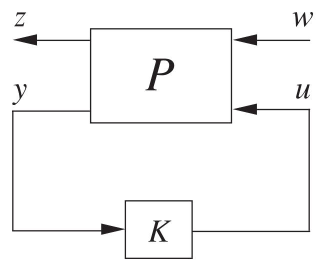

Fig 1. H^∞ control scheme.

The plant (2.1) can be written in the input/output form as

$$\begin{bmatrix} z(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} \begin{bmatrix} w(z) \\ u(z) \end{bmatrix}. \tag{2.2}$$

A feedback control law

$$u(z) = K(z)y(z) \tag{2.3}$$

generates the closed-loop transfer function $\Phi(z)$ from $w(z)$ to $z(z)$ given by

$$\Phi : = LF(P; K) := P_{11} + P_{12}K(I - P_{22}K)^{-1} P_{21}. \tag{2.4}$$

The objective is to find a control law (2.3) which internally stabilizes the closed-loop system of Fig 1, achieving the normalized norm bound of $\Phi(z)$, that is,

$$\|\Phi\|_\infty < 1. \tag{2.5}$$

Chain-Scattering Representation

Assuming that P_{21} is invertible, we have

$$w = P_{21}^{-1}(y - P_{22}u).$$

Substituting this relation in the first equation of (2.2) yields

$$z = (P_{12} - P_{11}P_{21}^{-1}P_{22})u + P_{11}P_{21}^{-1}y.$$

Therefore, if we write

$$G := CHAIN(P)$$

$$:= \begin{bmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix},$$

the relation (2.2) is alternatively represented as

$$\begin{bmatrix} z(z) \\ w(z) \end{bmatrix} = \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix} \begin{bmatrix} u(z) \\ y(z) \end{bmatrix}. \tag{2.6}$$

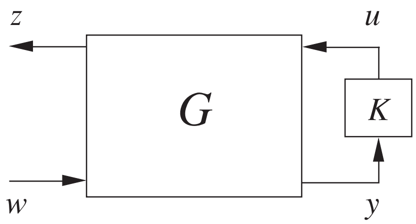

Fig 2. Chain-scattering representation of the system.

A feedback control law (2.3) applied to the chain-scattering representation of plant (2.6) generates the closed-loop transfer function $\Phi(z)$ given by

$$\Phi(z) := HM(G;K) := (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}. \tag{2.7}$$

The symbol *HM* stands for the *HoMographic Transformation*, while *LF* stands for *Linear Fractional Transformation.*[6] The properties of the transformation *HM* are listed up in the following lemmas, which are based on the work of Kimura.[5] Their proof is essentially the same as in the continuous-time case.

Lemma 2.1 *Properties* of *HM*

(i) *If* P_{21}^{-1} *exists,* $LF(P; K) = HM(CHAIN(P); K)$.

(ii) $HM(I;K) = K$.

(iii) $HM(G_1 ;HM(G_2 ;K)) = HM(G_1 G_2 ;K)$.

(iv) *If* G^{-1} *exists,* $HM(G;K) = F$ *implies* $K = HM(G^{-1}; F)$.

Next, we recall the following theorem from Kongprawechnon and Kimura.[8]

Theorem 2.2 *Under the assumptions (A1) and (A2), the normalized H^∞ control problem is solvable iff*

(i) there exists a solution $X \geq 0$ of the algebraic Riccati equation

$$X = A^TXA + C_1^TC_1 - F^T(D_z^TJD_z + B^TXB)F \tag{2.8}$$

such that

$$\hat{A}_G := A + BF \tag{2.9}$$

is stable,

(ii) there exists a solution $Y \geq 0$ of the algebraic Riccati equation

$$Y = AYA^T + B_1B_1^T + L(D_wJD_w^T - CYC^T)L^T \tag{2.10}$$

such that

$$\hat{A}_H \quad := A + LC \qquad (2.11)$$

is stable,

(iii)

$$\sigma(XY) < 1, \qquad (2.12)$$

(iv) there exists a nonsingular matrix E_z such that

$$D_z^T J D_z + B^T X B = E_z^T J' E_z, \qquad (2.13)$$

holds,

(v) there exists a nonsingular matrix E_w such that

$$D_W J D_w^T - CYC^T = E_W J'' E_w^T, \qquad (2.14)$$

holds, where

$$F \quad := \begin{bmatrix} F_w \\ F_u \end{bmatrix}$$

$$= -(D_z^T J D_z + B^T X B)^{-1}(D_r^T C_1 + B^T X A),$$

$$L \quad := \begin{bmatrix} L_z & L_u \end{bmatrix}$$

$$= -(B_1 D_c^T + AYC^T)(D_w J D_w^T - CYC^T)^{-1},$$

$$D_r \quad := \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}, \; D_c \quad := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix},$$

$$D_w \quad := \begin{bmatrix} -I & D_{11} \\ 0 & D_{21} \end{bmatrix}, \; D_z \quad := \begin{bmatrix} D_{11} & D_{12} \\ I & 0 \end{bmatrix},$$

$$D_u \quad = \begin{bmatrix} -D_{12} & 0 \\ 0 & I \end{bmatrix}, \; C \quad := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

$$B \quad := \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \; B_u \quad := \begin{bmatrix} B_2 & 0 \end{bmatrix}.$$

In that case, a desired controller is given by

$$K = HM(\Pi_{11}^{-1}; S), \qquad (2.15)$$

where

$$\Pi_{11}^{-1} \quad = QV_w^{-1}, \qquad (2.16)$$

$$Q \quad := \left[\begin{array}{c|cc} A + B_1 F_w + B_2 F_u & U[B_2 + L_z D_{12} - L_y] \\ \hline F_u & I & 0 \\ C_2 + D_{21} F_w & 0 & I \end{array}\right], (2.17)$$

$$U \quad := \left(I - YX\right)^{-1} \qquad (2.18)$$

and V_w is a nonsingular matrix satisfying

$$V_w^T J_{pq} V_w = \quad (B_u + LD_u)^T X (I - YX)^{-1} (B_u + LD_u) + D_u^T (D_w J D_w^T - CYC^T)^{-1} D_u \qquad (2.19)$$

and S is an arbitrary matrix in BH^∞.

Main Results

In this section, we will consider the closed-loop structure of an H^∞ controller. From equation (2.15), (2.17) and the cascade property of HM, we have

$$K \quad = HM(QV_w^{-1}; S) = HM(Q; HM(V_w^{-1}; S)). \quad (3.1)$$

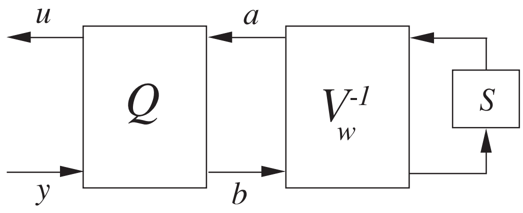

Fig 3. Chain-scattering representation of the controller.

Q given in (2.17) is described in the state-space as

$$\xi_{k+1} \;=\; \left(A + B_1 F_w + B_2 F_u\right)\xi_k + U\left(B_2 + L_z D_{12}\right)a_k - UL_Y b_k \;, \qquad (3.2a)$$

$$u_k \;=\; F_u \xi_k + a_k, \qquad (3.2b)$$

$$y_k \;=\; \left(C_2 + D_{21} F_w\right)\xi_k + b_k, \qquad (3.2c)$$

where ξ_k is the state of the controller. The controller (3.1) is obtained by introducing the relation

$$a_k \;=\; HM\left(V_w^{-1}; S\right) b_k \,.$$

The controller can be rewritten as

$$\xi_{k+1} \;=\; \left(A + B_1 F_w + B_2 F_u\right)\xi_k + U\left(\left(B_2 + L_z D_{12}\right) HM\left(V_w^{-1}; S\right) - L_y\right) b_k, \qquad (3.3a)$$

$$u_k \;=\; F_u \xi_k + HM\left(V_w^{-1}; S\right) b_k, \qquad (3.3b)$$

$$b_k \;=\; y_k - \left(C_2 + D_{21} F_w\right)\xi_k. \qquad (3.3c)$$

The block-diagram of the controller is illustrated in Fig 4. The meaning of the controller (3.3) will be discussed in the next section.

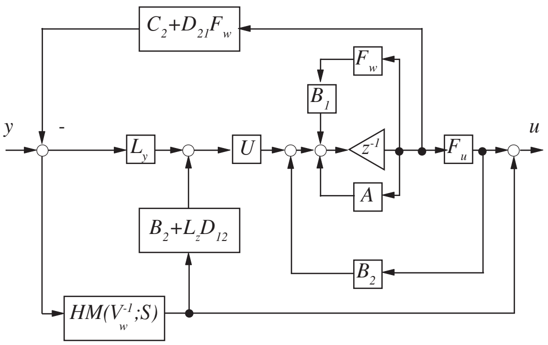

Fig 4. Block diagram of the H^∞ controller.

First, we will consider under the constraint imposed upon the controller to be strictly proper. As was recently shown by Mirkin *et al*[9], sampled-data control problems can always be formulated as discrete-time problems with strictly causal controllers. Hence, the consideration of strictly proper controllers does not lead to any loss of generality in most cases. To parameterize all strictly proper controllers is to extract the set of all strictly proper controllers from $K = HM(Q; HM(V_w^{-1}; S))$. In other words, one should find whether there exists a transfer matrix $S \in \boldsymbol{BH}^{\infty}_{(p+q)\times(p+q)}$ such that $K(\infty) = HM(Q; HM(V_w^{-1}; S))(\infty) = 0$. To this end, note that $Q(\infty)=I$. Due to (*ii*) of Lemma 2.1, $K(\infty) = HM(V_w^{-1}; S)(\infty)$. Let

$$V_w = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}.$$

To make $K(\infty) = HM(V_w^{-1}; S)(\infty)=0$, we should choose

$$S = HM\left(V_w; 0\right) = V_{12}V_{22}^{-1}, \qquad (3.4)$$

from (*iv*) of Lemma 2.1. Since $S \in \boldsymbol{BH}^{\infty}_{(p+q)\times(p+q)}$, hence

$$\left\| V_{12}V_{22}^{-1} \right\| < 1. \qquad (3.5)$$

Thus to adjust Theorem 2.2 to the case of strictly proper controller, one has to add (3.5) to the conditions of Theorem 2.2. From Lemma 2.1 of Ionescu *et al*[14], V_w can be chosen block lower{left triangular, that is,

$$V_w = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix},$$

which always satisfy (3.5).

Next we will consider the structure of the central controller. From $K = HM(Q; HM(V_w^{-1}; S)) = HM(Q; R)$, where $R := HM(V_w^{-1}; S)$, we have

$$S = HM\left(V_w; R\right) = V_{11}R\left(V_{21}R + V_{22}\right)^{-1}.$$

If we choose $S = 0$, then we have $R = 0$. Moreover, we obtain the so-called central controller, which is described as

$$\xi_k = A\xi_k + B_1\hat{\omega}_{0k} + B_2u_k - UL_y\left(y_k - C_2\xi_k - D_{21}\hat{\omega}_{0k}\right), \qquad (3.6a)$$

$$u_k = F_u\xi_k, \qquad (3.6b)$$

where

$$\hat{w}_{0k} = F_w\xi_k. \qquad (3.7)$$

The representation (3.6a)-(3.7) clarifies the observer structure of the central controller. Fig 5 illustrates the block diagram of this controller.

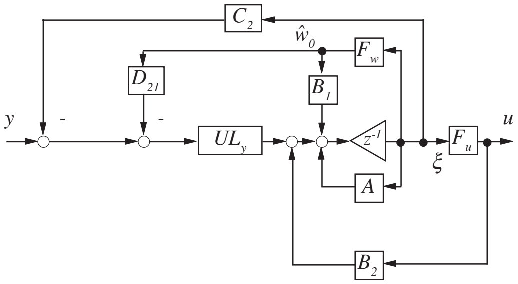

Fig 5. Block diagram of the central controller.

Stability Properties

In this section, we prove the stability of some closed-loop matrices based on the solutions to the Riccati equations (2.8) and (2.10), under the standing assumptions,

$$D_{11} = 0, \quad I - B_1^T XB_1 > 0, \quad I - C_1YC_1^T > 0. \qquad (4.1)$$

In this case,

$$D_z = \begin{bmatrix} 0 & D_{12} \\ I & 0 \end{bmatrix}, \quad D_w = \begin{bmatrix} -I & 0 \\ 0 & D_{21} \end{bmatrix}.$$

It follows that

$$E_z^T J' E_z = D_z^T J D_z + B^T XB$$
$$= \begin{bmatrix} B_1^T XB_1 - I & B_1^T XB_2 \\ B_2^T XB_1 & B_2^T XB_2 + D_{12}^T D_{12} \end{bmatrix},$$

$$E_w J'' E_w^T = D_w J D_w^T - CYC^T$$
$$= -\begin{bmatrix} C_1YC_1^T - I & C_1YC_2^T \\ C_2YC_1^T & D_{21}D_{21}^T + C_2YC_2^T \end{bmatrix}.$$

From the lemma 2.1 of Ionescu *et al*[14], then the condition (iv) of the Theorem 2.2 can be written as

$$(I - B_1^T XB_1) + B_1^T XB_2(B_2^T XB_2 + D_{12}^T D_{12})^{-1} B_2^T XB_1 > 0. \qquad (4.2)$$

and the condition (v) of the Theorem 2.2 can be written to be

$$(I - C_1YC_1^T) + C_1YC_2^T(D_{21}D_{21}^T + C_2YC_2^T)^{-1} C_2YC_1^T > 0. \qquad (4.3)$$

From the assumption (4.1) the inequalities (4.2) and (4.3) are both satisfied in this case. Now we summarize the result.

Corollary 4.1 *Under the assumptions (4.1), the* H^{∞} *control problem is solvable iff the following conditions hold.*

(i) There exists a solution $X \geq 0$ of the Riccati equation (2.8) that stabilizes $\hat{A}_G = A + BF$.

(ii) There exists a solution $Y \geq 0$ of the Riccati equation (2.10) that stabilizes $\hat{A}_H = A + LC$.

(iii) $\sigma(XY) < 1$

Under the assumption (4.1), the plant is now given by

$$x_{k+1} = Ax_k + B_1 w_k + B_2 u_k, \quad (4.4a)$$

$$z_k = C_1 x_k + D_{12} u_k, \quad (4.4b)$$

$$y_k = C_2 x_k + D_{21} w_k. \quad (4.4c)$$

Let $X \geq 0$ be a solution to the algebraic Riccati equation (2.8) and write

$$\psi(x_k) = x_k^T X x_k. \quad (4.5)$$

The differential of $\psi(x_k)$ along the trajectory of (4.4a) is calculated to be

$$\delta(\psi(x_k)) = \psi(x_{k+1}) - \psi(x_k) = x_{k+1}^T X x_{k+1} - x_k^T X x_k.$$

Using (2.8) and (4.4a), we obtain

$$\delta(\psi(x_k)) = \begin{bmatrix} w_k - F_w x_k \\ u_k - F_u x_k \end{bmatrix}^T M_F \begin{bmatrix} w_k - F_w x_k \\ u_k - F_u x_k \end{bmatrix} - z_k^T z_k + w_k^T w_k,$$

where

$$M_F = \begin{bmatrix} B_1^T X B_1 - I & B_1^T X B_2 \\ B_2^T X B_1 & B_2^T X B_2 + D_{12}^T D_{12} \end{bmatrix}$$

Taking a sum with respect to k from $k = 0$ to $k = T$ yields

$$x_T^T X x_T - x_0^T X x_0 = \sum_{k=0}^{T} \left(\| w_k \|^2 - \| z_k \|^2 \right) + \sum_{k=0}^{T} \left(\begin{bmatrix} w_k - F_w x_k \\ u_k - F_u x_k \end{bmatrix}^T M_F \begin{bmatrix} w_k - F_w x_k \\ u_k - F_u x_k \end{bmatrix} \right).$$

Taking $x_0 = 0$ gives

$$J(w,u;T) := \sum_{k=0}^{T} \left(\| w_k \|^2 - \| z_k \|^2 \right) = x_T^T X x_T - \sum_{k=0}^{T} \left(\begin{bmatrix} w_k - F_w x_k \\ u_k - F_u x_k \end{bmatrix}^T M_F \begin{bmatrix} w_k - F_w x_k \\ u_k - F_u x_k \end{bmatrix} \right).$$

This is a key identity from which various properties of H^∞ control can be extracted. Obviously, $J(w,u;T) \to \|w\|^2 - \|z\|^2$, as $T \to \infty$. The design objective is attained iff

$$J(w,u;\infty) > 0, \forall w \in L_2. \quad (4.6)$$

Actually, if we use the state feedback control law

$$u_0 = F_u x, \quad (4.7)$$

the design objective (4.6) is attained with possible equality due to the assumption (4.1) and $X \geq 0$.

Lemma 4.2 *Under the assumption* (4.1), *the state feedback (4.7) achieves the design objective for the plant (4.4a).*

On the other hand, the exogenous signal given by

$$w_0 = F_w x \quad (4.8)$$

represents the worst case for the controller, in the sense that it maximizes $J(w, u; \infty)$.

Let us focus on the central controller again. We can see that the representation (3.6a) clarifies the pseudo-state feedback structure of the central controller. The control law (3.6b) is just the replacement of x by ξ in (4.7). Hence, (3.7) represents the estimate of the state feedback control law. The most interesting feature of (3.6a) is that it assumes the exogenous signal w to be the worst. The signal $\hat{w}_0$ given by (3.7) represents the estimate of the worst one. In view of (4.4c), $b = y - C_2\xi - D_{21}\hat{w}_0$ represents the innovation assuming $w = \hat{w}_0$. The observer gain is given by UL_y.

EXAMPLES

In this section, a collection of simple examples is given, in order to get an idea of the structure of H^∞ control.

Example 5.1 Consider a first-order plant

$$x_{k+1} = ax_k + b_1 w_k + b_2 u_k,$$

$$z_k = c_1 x_k + u_k,$$

$$y_k = c_2 x_k + w_k,$$

where all the quantities in these expressions are scalar. The assumption (A1) implies that

$$a < 1 \text{ or } b_2 c_2 \neq 0.$$

The assumption (A2) implies that

$$\left((a - b_2 c_1)^2 - 1 \right)\left((a - b_1 c_2)^2 - 1 \right) \neq 0.$$

The Riccati equation (2.8) becomes in this case

$$\left((a - b_2 c_1)^2 - 1 \right) X - (b_2^2 - b_1^2) X^2 = 0.$$

The stabilizing solution $X \geq 0$ exists iff $\beta_c > 0$, and is given by

$$X = \begin{cases} 0, & \text{if } \alpha_c < 1, \\ \dfrac{\alpha_c^2 - 1}{\beta_c}, & \text{if } \alpha_c > 1, \end{cases}$$

where

$$\alpha_c := a - b_2 c_1, \quad \beta_c := b_2^2 - b_1^2,$$

and the stabilized matrix is given by

$$A + BF = \begin{cases} \alpha_c, & \text{if } \alpha_c < 1, \\ \dfrac{1}{\alpha_c}, & \text{if } \alpha_c > 1. \end{cases}$$

It is important to notice that the role of F is to bring the closed-loop pole at the mirror image of a-b_2c_1, the zero of $P_{12}(z)$. Dually, the Riccati equation (2.10) becomes in this case

$$\left((a-b_1c_2)^2-1\right)Y-(c_2^2-c_1^2)Y^2=0.$$

The stabilizing solution Y exists iff $\beta_o > 0$, and is given by

$$Y=\begin{cases} 0, & \text{if} \quad \alpha_0<1, \\ \dfrac{\alpha_0^2-1}{\beta_0}, & \text{if} \quad \alpha_0>1, \end{cases}$$

where

$$\alpha_o := a-b_1c_2, \quad \beta_o := c_2^2-c_1^2,$$

and the stabilized matrix (2.11) is given by

$$A+LC=\begin{cases} \alpha_o, & \text{if } \alpha_o<1, \\ \dfrac{1}{\alpha_o}, & \text{if } \alpha_o>1. \end{cases}$$

Example 5.2 This example is the so-called two-block case. Numerical computations are performed on the control design software MATLAB. We consider the following second-order system

$$x_{k+1}=\begin{bmatrix} 0.1 & 0 \\ 0 & 2 \end{bmatrix}x_k+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u_k,$$

$$z_k=\begin{bmatrix} 0 & 1.7 \end{bmatrix}x_k+u_k,$$

$$y_k=\begin{bmatrix} 0 & 4.2102 \end{bmatrix}x_k+\begin{bmatrix} 0 & 0.7988 \end{bmatrix}w_k.$$

The solution of (2.8) and (2.10) are given respectively by

$$X=0, \quad Y=\begin{bmatrix} 0 & 0 \\ 0 & 0.1205 \end{bmatrix}>0.$$

The matrices $\hat{A}_G$ in (2.9) and $\hat{A}_H$ in (2.11) are given respectively by

$$\hat{A}_G=\begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad \hat{A}_H=\begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix},$$

which satisfy the conditions of Theorem 2.2. In this case, the central controller is given by

$$\xi_{k+1}=\begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}\xi_k+\begin{bmatrix} 0 \\ 0.1801 \end{bmatrix}v_k,$$

$$u_k=\begin{bmatrix} 0 & -1.7 \end{bmatrix}\xi_k-0.1973v_k,$$

$$v_k=y_k-\begin{bmatrix} 0 & 4.2102 \end{bmatrix}\xi_k.$$

Remark: This is the case that $A-B_2D_{12}^{-1}C_1$ is stable. Therefore, Condition (i) of Theorem 2.2 is unnecessary. Also, Condition (iii) holds automatically and Condition (iv) can be checked easily.

Example 5.3 Finally, we consider another second-order system. This is an example of four-block case.

$$x_{k+1}=\begin{bmatrix} 0 & 0.9665 \\ 1.1387 & 0 \end{bmatrix}x_k+\begin{bmatrix} 0 & 0 \\ 1.2828 & 0 \end{bmatrix}w_k+\begin{bmatrix} 0 \\ 6.0231 \end{bmatrix}u_k,$$

$$z_k=\begin{bmatrix} 0.1884 & 0 \\ 0 & 0 \end{bmatrix}x_k+\begin{bmatrix} 0 \\ 0.7821 \end{bmatrix}u_k,$$

$$y_k=\begin{bmatrix} 3.3510 & 0 \end{bmatrix}x_k+\begin{bmatrix} 0 & 5.4403 \end{bmatrix}w_k.$$

Then we obtain

$$X=\begin{bmatrix} 0.0521 & 0 \\ 0 & 0.0486 \end{bmatrix}>0,$$

and

$$Y=\begin{bmatrix} 3.4473 & 0 \\ 0 & 3.6908 \end{bmatrix}>0.$$

satisfy all conditions of Theorem 2.2. In this case, the central controller is given by

$$\xi_{k+1}=\begin{bmatrix} 0 & 0.9665 \\ 0.2993 & 0 \end{bmatrix}\xi_k-\begin{bmatrix} 0 \\ 0.0549 \end{bmatrix}b_k,$$

$$u_k=\begin{bmatrix} -0.1433 & 0 \end{bmatrix}\xi_k-0.0263b_k,$$

$$b_k=y_k-\begin{bmatrix} 3.351 & 0 \end{bmatrix}\xi_k.$$

Conclusion

The closed-loop structure of discrete-time H^∞ control has been discussed. The existence condition for a strictly proper H^∞ controller for discrete-time systems has also been derived. We believe that the result derived in this paper may be a useful tool in solving various control problems with the H^∞ performance measure.

Acknowledgements

The authors deeply appreciate the support from the Thailand Research Fund through the postdoctoral fund program.

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