The Relationship between the Nine-Point Circle, the Circumscribed Circle and the Inscribed Circle of Archimedes' Triangle

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Abstract

The purpose of this research was to study the relationship between the Nine-Point Circle and the Circumscribed circle of Archimedes' triangle, and the relationship between the Nine-Point circle and the Inscribed circle of Archimedes' triangle. The results were that the radius of the Nine-Point Circle of Archimedes' triangle is half the radius of the Circumscribed circle of Archimedes' triangle, and the Nine-Point circle and the Inscribed circle of Archimedes' triangle touch internally.

Keywords: Archimedes' triangle, Nine-Point circle, Circumscribed circle of triangle, Inscribed circle of triangle

Introduction

The Swiss mathematician, Leonhard Euler (1707-1783), discovered the Nine-Point Circle of a triangle, and that its circumference passes through nine points, with the first three points being the midpoint of the triangle's sides, and the three points originating at the perpendicular line from the vertex to meet the opposite sides of the triangle, and the other three points being the midpoint of the distance between the orthocenter and the vertex of the triangular angles. (Davis, 2002). Karl Feuerbach (1800-1834), a German mathematician, also described this, in detail, Thus, this is also called the Feuerbach Circle or Euler Circle.

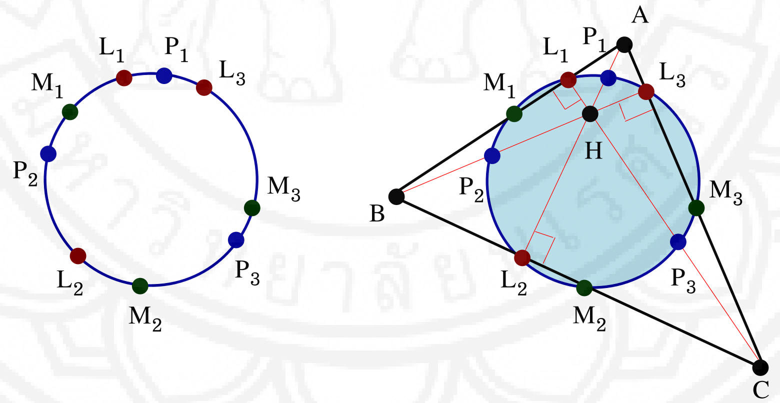


Figure 1 the Nine-Point Circle

Figure 1; H is the orthocenter of triangle ABC, a circle with its circumference passing through nine points which are the midpoints of triangle's sides (M_1, M_2, M_3), with the points originating at the perpendicular line from the vertex to the opposite sides (L_1, L_2, L_3) and the midpoint of the orthocenter and the vertex of the triangles (P_1, P_2, P_3) call the Nine-Point Circle of the ABC triangle.

The Greek mathematician, Archimedes (287-212 B.C.), discovered that the parabolic segment of the parabolic curve, and the cord, link on two points on their parabolic curve, then, from lines touching the parabolic curve at the end of the cords along to the outside parabolic segment, to form an Archimedes'

Triangle. Thus, the base of triangle is the parabolic segment cord and the other two sides are lines touching the parabolic curve at the end of the cords. (Erbas, 2000; Woltermann, 2014)

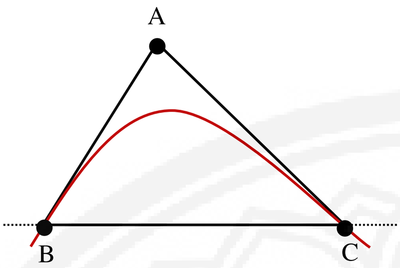


Figure 2.1 Archimedes' Triangle

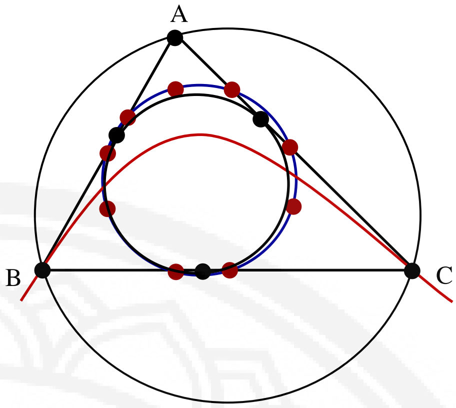


Figure 2.2 the Nine-Point Circle, the Circumscribed Circle and the Inscribed Circle of Archimedes' Triangle

Figure 2.1; An Archimedes' Triangle ABC from the line meeting the parabolic curve at point A and B, side AC and BC from the lines attaching the parabolic curve at A and B and meeting at point C.

All of the Nine-Point Circle, the Circumscribed Circle and the Inscribed Circle have structural relationships with Archimedes' Triangle (Figure 2.2) thus, the aims of this research were to describe relationship of the Nine-Point Circle, the Circumscribed Circle and the Inscribed Circle of Archimedes' Triangle by using the analytical geometry method to expand the mathematic knowledge for developing other subjects.

Methods and Materials

This research used the following procedures; at first review basic knowledge study about definition, property, co-ordinates, composition and structure of the Nine-Point Circle, the Circumscribed Circle and the Inscribed Circle of Archimedes' Triangle by using the analytical geometry method, and then find radius of the Nine-Point Circle and the Circumscribed Circle for describe relationship of them by using Euclidian geometry and rectangular analytic geometry, solution of quadratic equation, calculus and ratio compare method, finally find distance from center of the Nine-Point Circle to the Inscribed Circle for compare this with the difference of radius of the Nine-Point Circle and radius of the Inscribed Circle to describe relationship of them by using property of circle and Euclidian geometry with trigonometry proof.

Results

1. Basic Knowledge

1.1 Archimedes' Triangle

Definition 1 Let parabolic segment is area that's enclosed with parabolic curve and cord link on two points on their parabolic curve, Archimedes' Triangle is the triangle its base is parabolic segment cord and other two sides are lines touch parabolic curve at end of the cords.

1.2 Archimedes' Triangle in Rectangular Co-ordinate System

Determine Archimedes' Triangle in Rectangular Co-ordinate system is a result of line $y = mx + c$, where m and c are real number and not equal to 0 simultaneously meet parabolic curve $y = ax^2$, where a is real number and not equal to 0 at point B and C. Cord BC is a base of this triangle, side AB and AC originated from two tangent line of parabolic curve at B and C, point of intersection is A.

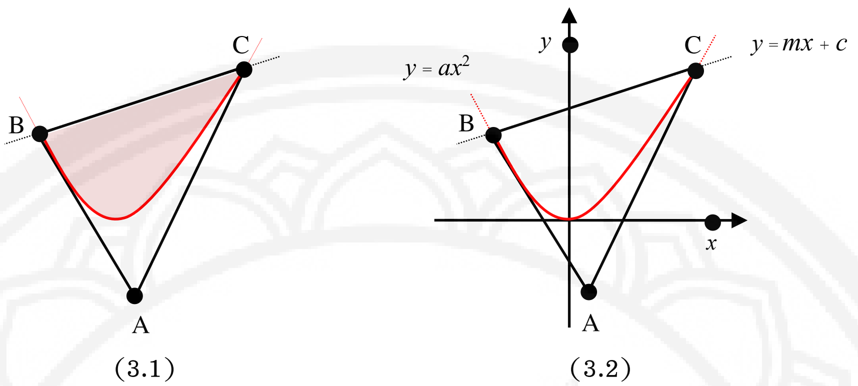


Figure 3 Archimedes' Triangle in Rectangular Co-ordinate System

Manoosilp (2014) and Rimcholakarn (2017) studied to co-ordinate vertex of Archimedes' Triangle ABC by solving equation for intersection point of line BC and parabolic curve and intersection point of line AB and AC. then, they found;

$$A\left(\frac{m}{2a}, -c\right)$$

$$B\left(\frac{m-\sqrt{m^2+4ac}}{2a}, \frac{m^2-m\sqrt{m^2+4ac}+2ac}{2a}\right)$$

$$C\left(\frac{m+\sqrt{m^2+4ac}}{2a}, \frac{m^2+m\sqrt{m^2+4ac}+2ac}{2a}\right)$$

1.3 Orthocenter of Archimedes' Triangle

Definition 2 The Orthocenter of a triangle is the intersection point originating from a perpendicular line from the vertex meet the opposite triangle sides. (Dunham, 1998)

Archimedes' Triangle ABC is a result of line $y = mx + c$, meet parabolic curve $y = ax^2$, at point B and C. Let L_1, L_2 and L_3 is the point originated the perpendicular line from the vertex meet the side AB, BC and AC respectively. H (Orthocenter) is intersection point of line L_1, L_2 and L_3.

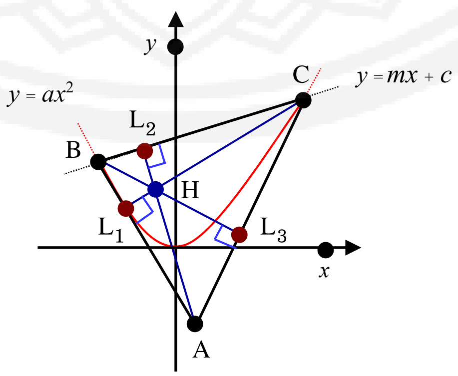


Figure 4 Orthocenter of Archimedes' Triangle

Finding Co-ordinate of Orthocenter of Archimedes' Triangle

Figure 4; Line CL_1 perpendicular to AB, from the derivative of $y = ax^2$ and substitute with $\frac{m-\sqrt{m^2+4ac}}{2a}$ thus, the slop of line AB = $m-\sqrt{m^2+4ac}$,

therefore, slop of line CL_1 = $-[\frac{1}{m-\sqrt{m^2+4ac}}]$

Co-ordinate of point C at $(\frac{m+\sqrt{m^2+4ac}}{2a}, \frac{m^2+m\sqrt{m^2+4ac}+2ac}{2a})$

thus, linear equation CL_1 is

$$y-[\frac{m^2+m\sqrt{m^2+4ac}+2ac}{2a}] = -[\frac{1}{m-\sqrt{m^2+4ac}}][x-(\frac{m+\sqrt{m^2+4ac}}{2a})]$$

$$y = [\frac{m^2+m\sqrt{m^2+4ac}+2ac}{2a}]-[\frac{x}{m-\sqrt{m^2+4ac}}]+[\frac{m+\sqrt{m^2+4ac}}{(2a)(m-\sqrt{m^2+4ac})}] \quad \text{--- (1)}$$

From line BL_3 perpendicular to AC the slop of line AC = $m+\sqrt{m^2+4ac}$,

therefore, slop of line BL_3 = $-[\frac{1}{m+\sqrt{m^2+4ac}}]$

Co-ordinate of point B at $(\frac{m-\sqrt{m^2+4ac}}{2a}, \frac{m^2-m\sqrt{m^2+4ac}+2ac}{2a})$,

thus, linear equation CL_1 is

$$y-[\frac{m^2-m\sqrt{m^2+4ac}+2ac}{2a}] = -[\frac{1}{m+\sqrt{m^2+4ac}}][x-(\frac{m-\sqrt{m^2+4ac}}{2a})]$$

$$y = [\frac{m^2-m\sqrt{m^2+4ac}+2ac}{2a}]-[\frac{x}{m+\sqrt{m^2+4ac}}]+[\frac{m-\sqrt{m^2+4ac}}{(2a)(m+\sqrt{m^2+4ac})}] \quad \text{--- (2)}$$

Finding co-ordinate of point H from intersection line BL_3 and CL_1

consider (1) and (2) result from (2)-(1):

$$(\frac{-2\sqrt{m^2+4ac}}{-4a})x = \frac{-2m\sqrt{m^2+4ac}}{2a}+\frac{(-4m)\sqrt{m^2+4ac}}{(2a)(m^2-m^2-4ac)}$$

$$(\frac{\sqrt{m^2+4ac}}{2ac})x = \frac{-m\sqrt{m^2+4ac}}{a}+\frac{m\sqrt{m^2+4ac}}{2a^2c}$$

$$(\frac{\sqrt{m^2+4ac}}{2ac})x = \frac{-2ac(m\sqrt{m^2+4ac})+m\sqrt{m^2+4ac}}{2a^2c}$$

$$x = \frac{m-2acm}{a}$$

substitute x in (2)

$$y = \left[\frac{m^2+m\sqrt{m^2+4ac}+2ac}{2a}\right]-\left[\frac{\frac{m-2acm}{a}}{m-\sqrt{m^2+4ac}}\right]+\left[\frac{m+\sqrt{m+4ac}}{(2a)(m-\sqrt{m^2+4ac})}\right]$$

$$= \frac{(m^2+m\sqrt{m^2+4ac}+2ac)(m-\sqrt{m^2+4ac})-2(m-2acm)+(m+\sqrt{m^2+4ac})}{(2a)(m-\sqrt{m^2+4ac})}$$

$$= \frac{(2ac-1)(m-\sqrt{m^2+4ac})}{(2a)(m-\sqrt{m^2+4ac})}$$

$$y = \frac{2ac-1}{2a}$$

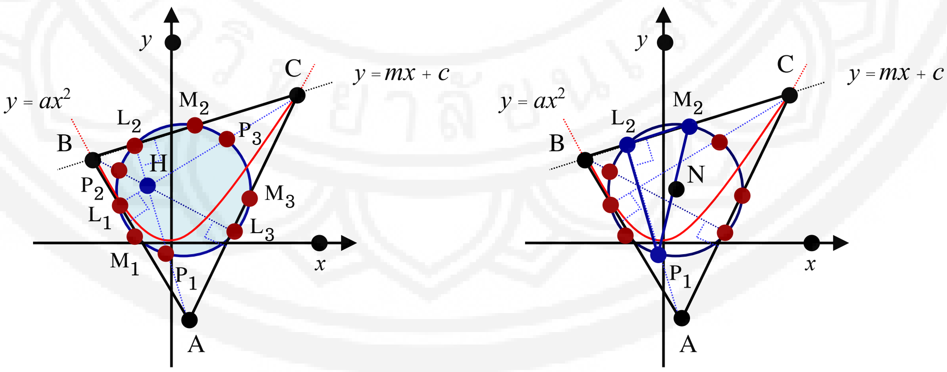
then, the co-ordinate of orthocenter of Archimedes' Triangle at H $\left(\frac{m-2acm}{a}, \frac{2ac-1}{2a}\right)$

2. The Nine-Point Circle and the Circumscribed Circle of Archimedes' Triangle

Definition 3 the Nine-Point Circle of Archimedes' triangle is the circle that its circumference pass through the three midpoints of triangle sides, then three points originated perpendicular line from vertex meet opposite triangle sides and three midpoint of distance between orthocenter and vertex of Archimedes' Triangle.

2.1 Radius of the Nine-Point Circle of Archimedes' Triangle

Determine the circumference of the Nine-Point circle of Archimedes' Triangle ABC pass through point L_1, P_2, L_2, M_2, P_3, M_3, L_3, P_1 and M_1 respectively, where M_1, M_2 and M_3 is the midpoint of side AB, BC and AC, point L_1, L_2 and L_3 originated the perpendicular line from vertex meet side AB, BC and AC, point H is orthocenter and point P_1, P_2, and P_3 is the midpoint of the line AH, BH and CH respectively.



(5.1) (5.2)

Figure 5 Radius and Area of the Nine-Point Circle of Archimedes' Triangle

Figure 5.2; Point M_2, L_2 and P_1 are on the circumference and $\angle M_2L_2P_1$ is a right angle, thus, $\angle M_2L_2P_1$ is an angle in semicircle and line M_2P_1 is the diameter of the Nine-Point Circle.

Finding Co-ordinate of Point M_2 (midpoint of side BC)

M_2 is midpoint of side BC, thus, co-ordinate of M_2 is

$$x = \frac{\frac{m-\sqrt{m^2+4ac}}{2a}+\frac{m+\sqrt{m^2+4ac}}{2a}}{2}$$

$$= \frac{m}{2a}$$

$$y = \frac{\frac{m^2-m\sqrt{m^2+4ac}+2ac}{2a}+\frac{m^2+m\sqrt{m^2+4ac}+2ac}{2a}}{2}$$

$$= \frac{m^2+2ac}{2a}$$

then, co-ordinate of the midpoint of side BC at $M_2\left(\frac{m}{2a}, \frac{m^2+2ac}{2a}\right)$

Finding Co-ordinate of Point P_1 (midpoint of line AH)

P_1 is the midpoint of the distance from A $\left(\frac{m}{2a}, -c\right)$ to H $\left(\frac{m-2acm}{a}, \frac{2ac-1}{2a}\right)$,

thus, the co-ordinate of point P_1 is

$$x = \frac{\frac{m}{2a}+\frac{m-2acm}{a}}{2}, \qquad y = \frac{(-c)+\frac{2ac-1}{2a}}{2}$$

$$= \frac{3m-4acm}{4a}, \qquad = -\frac{1}{4a}$$

then, co-ordinate of midpoint of line AH at $P_1\left(\frac{3m-4acm}{4a}, -\frac{1}{4a}\right)$

Finding the Co-ordinate of Center of the Nine-Point Circle of Archimedes' Triangle

Let point N is the center of the Nine-Point Circle of Archimedes' Triangle (midpoint of line M_2P_1), thus, co-ordinate of point N is

$$x = \frac{\frac{m}{2a}+\frac{3m-4ac}{4a}}{2}, \qquad y = \frac{\frac{m^2+2ac}{2a}+\left(-\frac{1}{4a}\right)}{2}$$

$$= \frac{5m-4ac}{8a}, \qquad = \frac{2m^2+4ac-1}{8a}$$

then, the co-ordinate of center of the Nine-Point Circle at $P_1\left(\frac{5m-4ac}{8a}, \frac{2m^2+4ac-1}{8a}\right)$

Finding the length of Radius of the Nine-Point Circle of Archimedes' Triangle

Let r_n = P_1N is the radius of the Nine-Point Circle of Archimedes' Triangle

$$r_n = |\ P_1N| = \sqrt{[(\frac{5m-4acm}{8a})-(\frac{3m-4acm}{4a})]^2+[(\frac{2m^2+4ac-1}{8a})-(\frac{-1}{4a})]^2}$$

$$= \sqrt{[\frac{5m-4acm-6m+8acm}{8a}]^2+[\frac{2m^2+4ac-1+2}{8a}]^2}$$

$$= \sqrt{\frac{16a^2c^2m^2-8acm^2+m^2}{64a^2}+\frac{4m^4+4m^2+8ac+16a^2c^2+16acm^2+1}{64a^2}}$$

$$= \sqrt{\frac{4m^4+5m^2+8acm^2+8ac+16a^2c^2+16a^2c^2m^2+1}{64a^2}}$$

then, the length of Radius of the Nine-Point Circle of Archimedes' Triangle is equal to

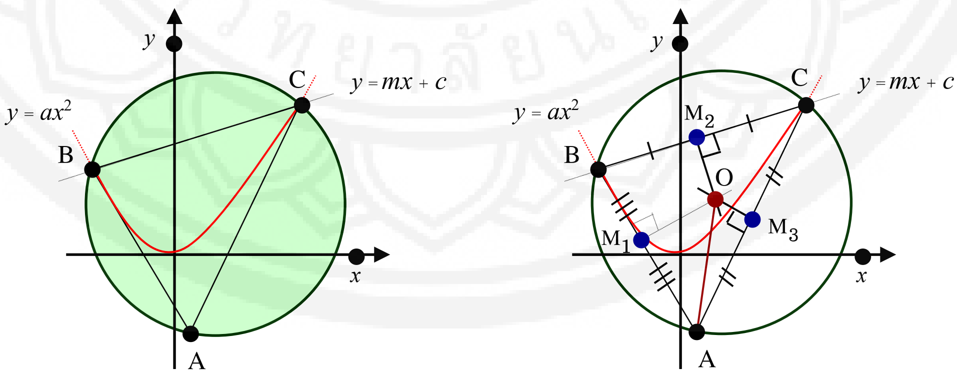
$$\sqrt{\frac{4m^4+5m^2+8acm^2+8ac+16a^2c^2+16a^2c^2m^2+1}{64a^2}} \text{ unit}$$

2.2 Radius of the Circumscribed Circle of Archimedes' Triangle

Definition 4 the Circumscribed circle of the triangle is the circle passing through the three vertices of the triangle. (Yiu, 2001)

Definition 5 C the ircumcenter of a triangle is the intersection point of the perpendicular bisectors of the three sides of the triangle. (Dunham, 1998)

Determine Archimedes' Triangle ABC is a result of line $y = mx + c$, meet the parabolic curve $y = ax^2$, at point B and C. Let M_1, M_2 and M_3 is the midpoint of side AB, BC and AC respectively. O is the intersection point of the perpendicular bisectors of the three sides of Archimedes' Triangle ABC.



(6.1) (6.2)

Figure 6 Radius and Area of the Circumscribed Circle of Archimedes' Triangle

Finding Co-ordinate of Circumcenter of Archimedes' Triangle

Co-ordinate of midpoint of side BC is $M_2\left(\frac{m}{2a}, \frac{m^2+2ac}{2a}\right)$ and slop of BC = m,

from the M_2O perpendicular to BC, therefore, slop of line M_2O = $-[\frac{1}{m}]$

thus, linear equation of M_2O is

$$y-[\frac{m^2+2ac}{2a}] = -[\frac{1}{m}][x-\frac{m}{2a}]$$

$$y = [\frac{m^2+2ac}{2a}]-[\frac{x}{m}]+[\frac{1}{2a}] \qquad \text{------ (3)}$$

Point A $\left(\frac{m}{2a}, -c\right)$ and C $\left(\frac{m+\sqrt{m^2+4ac}}{2a}, \frac{m^2+m\sqrt{m^2+4ac}+2ac}{2a}\right)$ are

the end point of side AC and M_3 is the midpoint of side AC, thus, the co-ordinate of point M_3 is

$$x = \frac{\frac{m}{2a}+\frac{m+\sqrt{m^2+4ac}}{2a}}{2}, \qquad y = \frac{(-c)+\frac{m^2+m\sqrt{m^2+4ac}+2ac}{2a}}{2}$$

$$= \frac{2m+\sqrt{m+4ac}}{4a}, \qquad = \frac{m^2+m\sqrt{m^2+4ac}}{4a}$$

then, co-ordinate of midpoint of side AC at $M_3\left(\frac{2m+\sqrt{m+4ac}}{4a}, \frac{m^2+m\sqrt{m^2+4ac}}{4a}\right)$

$$\text{the slop of side AC} = \frac{\frac{m^2+m\sqrt{m^2+4ac}+2ac+2ac}{2a}}{\frac{m+m\sqrt{m^2+4ac}-m}{2a}}$$

$$= m+\sqrt{m^2+4ac}$$

the line M_3O perpendicular to AC, therefore, slop of line M_3O = $-[\frac{1}{m+\sqrt{m^2+4ac}}]$

The linear equation of line M_3O is

$$y-[\frac{m^2+m\sqrt{m^2+4ac}}{4a}] = -[\frac{1}{m+\sqrt{m^2+4ac}}][x-(\frac{2m+\sqrt{m^2+4ac}}{4a})]$$

$$y = [\frac{m^2+m\sqrt{m^2+4ac}}{4a}]-[\frac{x}{m+\sqrt{m^2+4ac}}]+[\frac{2m+\sqrt{m^2+4ac}}{(m+\sqrt{m^2+4ac})(4a)}] \qquad \text{----- (4)}$$

Point O is the Circumcenter of a triangle and also the intersection point of line M_2O and M_3O, thus, from (3) and (4) the co-ordinate of point O is

$$[\frac{m^2+2ac}{2a}]-[\frac{x}{m}]+[\frac{1}{2a}] = [\frac{m^2+m\sqrt{m^2+4ac}}{4a}]-[\frac{x}{m+\sqrt{m^2+4ac}}]+[\frac{2m+\sqrt{m^2+4ac}}{(m+\sqrt{m^2+4ac})(4a)}]$$

$$[\frac{m-m-\sqrt{m^2+4ac}}{(m+\sqrt{m+4ac})(m)}]x = \frac{-(4ac+1)\sqrt{m^2+4ac}}{(m+\sqrt{m^2+4ac})(4a)}$$

$$x = \frac{-[(4ac+1)\sqrt{m^2+4ac}][m+\sqrt{m^2+4ac}][m]}{(m+\sqrt{m^2+4ac})(4a)(-\sqrt{m^2+4ac})}$$

$$= \frac{4acm+m}{4a}$$

substitute x in (3)

$$y = [\frac{m^2+2ac}{2a}]-[\frac{\frac{4acm+m}{4a}}{m}]+[\frac{1}{2a}]$$

$$= \frac{2m^3+4acm-4acm-m+2m}{4am}$$

$$= \frac{2m^2+1}{4a}$$

then, the co-ordinate of circumcenter of Archimedes'Ttriangle at $M_3\left(\frac{4acm+m}{4a}, \frac{2m^2+1}{4a}\right)$

Finding length of Radius of the Circumscribed Circle of Archimedes' Triangle

Let R = AO is radius of the Circumscribed circle of Archimedes' triangle

$$R = |AO| = \sqrt{[(\frac{4acm+m}{4a})-(\frac{m}{2a})]^2+[(\frac{2m^2+1}{4a})-(-c)]^2}$$

$$= \sqrt{[\frac{4acm-m}{4a}]^2+[\frac{2m^2+4ac+1}{4a}]^2}$$

$$= \sqrt{(\frac{16a^2c^2m^2-8acm^2+m^2}{16a^2})+(\frac{4m^4+4m^2+8ac+16a^2c^2+16acm^2+1}{16a^2})}$$

$$= \sqrt{\frac{4m^4+5m^2+8acm^2+8ac+16a^2c^2+16a^2c^2m^2+1}{16a^2}}$$

then, length of Radius of the Circumscribed Circle of Archimedes' Triangle is equal to

$$\sqrt{\frac{4m^4+5m^2+8acm^2+8ac+16a^2c^2+16a^2c^2m^2+1}{16a^2}} \quad \text{unit}$$

3. The Relationship between the Radius of the Nine-Point Circle and the Circumscribed Circle of Archimedes' Triangle

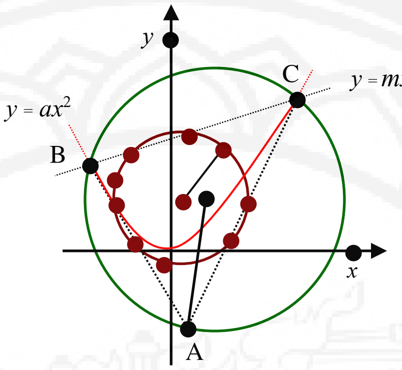


Figure 7 Relationship between the Nine-Point circle and the Circumscribed Circle of Archimedes' triangle

From figure 7; let

r_n is radius of the Nine-Point circle of Archimedes' triangle

R is radius of the Circumscribed circle of Archimedes' triangle

the ratio of the radius of the Nine-Point circle of Archimedes' triangle and radius of the Circumscribed circle of Archimedes' triangle is

$$\frac{r_n}{R} = \frac{(\pi)\sqrt{\dfrac{4m^4+5m^2+8acm^2+8ac+16a^2c^2+16a^2c^2m^2+1}{64a^2}}}{(\pi)\sqrt{\dfrac{4m^4+5m^2+8acm^2+8ac+16a^2c^2+16a^2c^2m^2+1}{16a^2}}}$$

$$= \frac{1}{2}$$

then, the radius of the Nine-Point Circle of Archimedes' Triangle is half of the radius of the Circumscribed Circle of Archimedes' triangle

Remark

Let r and R is the radius of the Nine-Point Circle and the radius of the Circumscribed Circle of Archimedes' Triangle respectively. the radius of the Nine-Point Circle of Archimedes' Triangle is half of the radius of the Circumscribed Circle of Archimedes' Triangle, so that, we can say $R = 2r$

4. The Nine-Point Circle and the Inscribed Circle of Archimedes' Triangle

Definition 6 the Inscribed Circle of triangle is the circle that its circumference touches three sides of triangle. The Incenter of a triangle is intersection point of the three angles bisectors of triangle.

4.1 Distance from Circumcenter to Incenter of Archimedes' Triangle

Determine point O and I is the circumcenter and incenter of Archimedes' Triangle ABC respectively, point Q_1, Q_2 and Q_3 is the point of tangency of the circumference of the Inscribed circle, line IQ_1, IQ_2 and IQ_3 are the radius of the Inscribed Circle and line OI is the distance from the circumcenter to the incenter of the Archimedes' Triangle.

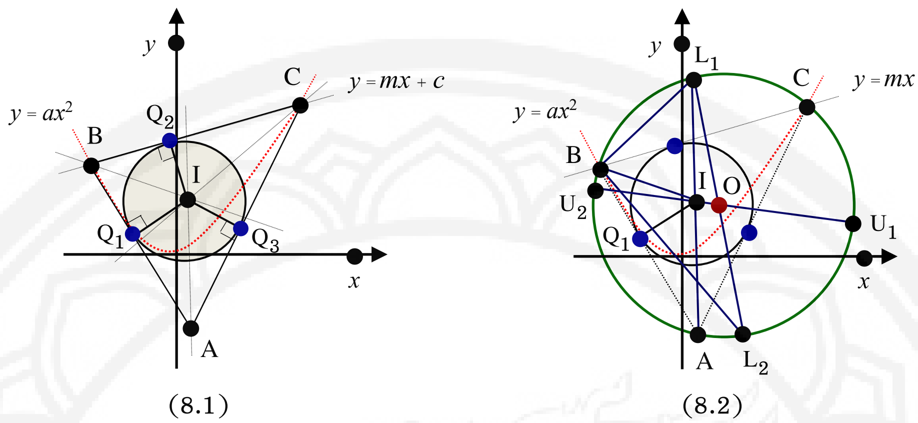


Figure 8 Relationship between the Nine-Point circle and the Inscribed circle of Archimedes' triangle

Finding distance from Circumcenter to Incenter of Archimedes' Triangle

Consider Figure 8.2; let R is radius of the Circumscribed Circle of Archimedes' Triangle

r is radius of the Inscribed Circle of Archimedes' Triangle

d is distance from point O to I

Draw a line from point A through I meet the circumference of the Circumscribed Circle at point L_1, us, line AL_1 is bisector of $\angle BAC$, so that L_1 is midpoint of curve BC

Draw a line from point L_1 through O meet the circumference of the Circumscribed Circle at point L_2, thus, line L_1L_2 is diameter of the Circumscribed Circle

Q_1 is point of tangency, thus, IQ_1 is the perpendicular line to side AB at point Q_1 and line $IQ_1 = r$ is radius of the Inscribed Circle

Triangle AQ_1I and BL_1L_2 are similar triangle because

$\angle AQ_1I = \angle BL_1L_2$ (both are right angle)

$\angle Q_1AI = \angle BL_2L_1$ (both angle on curve BL_1 are equal)

$\angle AIQ_1 = \angle BL_1L_2$ (if two pairs of two triangles are equal, so is the third angle)

From similar triangle AQ_1I and BL_1L_2, the ratio

$$IQ_1 : BL_1 = AI : L_1L_2$$

$$(IQ_1)(L_1L_2) = (AI)(BL_1)$$

$$2Rr = (AI)(BL_1) \qquad \text{----------- (5)}$$

Draw a line BI, let $\alpha = \angle BAC$ and $\beta = \angle ABC$

Consider triangle ABI, $\angle BIL_1 = \dfrac{\alpha}{2} + \dfrac{\beta}{2}$ and triangle BL_1I, $\angle IBL_1 = \dfrac{\alpha}{2} + \dfrac{\beta}{2}$,

thus, $\angle BIL_1 = \angle IBL_1$, BIL_1 is isosceles triangle and side $BL_1 = IL_1$

substitute IL_1 in (5) therefore, $2Rr = (AI)(IL_1)$

Draw a line from point O through I meet the circumference of the Circumscribed Circle at point U_1 and draw a line from point I through O meet the circumference of the Circumscribed Circle at point U_2, thus, line U_1U_2 is diameter of the Circumscribed Circle and $U_1U_2 = 2R$

From $(U_1I)(U_2I) = (AI)(IL_1) = 2Rr$, thus

$$(R+d)(R-d) = 2Rr$$
$$R^2 - d^2 = 2Rr$$
$$d^2 = R(R-2r)$$
$$(OI)^2 = R^2 - 2Rr \qquad \text{-------------} (6)$$

then, the distance from the circumcenter to the incenter of Archimedes' Triangle is in the form $(OI)^2 = R^2 - 2Rr$

4.2 Distance from Incenter to Orthocenter of Archimedes' Triangle

Determine point I and H is the incenter and orthocenter of the Archimedes' Triangle ABC respectively, line IH is the distance from the incenter to the the orthocenter of Archimedes' Triangle.

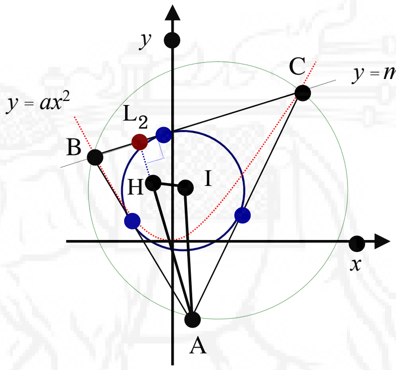


Figure 9 incenter and orthocenter of Archimedes' Triangle

Finding the distance from the Incenter to the Orthocenter **of the the Archimedes' Triangle**

Let R is radius of the Circumscribed Circle of Archimedes' Triangle

r is radius of the Inscribed Circle of Archimedes' Triangle

α, β and θ is $\angle BAC$, $\angle ABC$ and $\angle ACB$ respectively

From $\angle HAC = 90^\circ - \theta$ and $\angle IAC = \frac{\alpha}{2}$,

thus, $\angle HAI = 90^\circ - \theta - \frac{\alpha}{2} = \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\theta}{2} - \frac{2\theta}{2} - \frac{\alpha}{2} = \frac{\beta - \theta}{2}$

Line $AH = 2R\cos\alpha$ and $AI = 4R\sin(\frac{\beta}{2})\sin(\frac{\theta}{2})$

Consider triangle HAI, from the Law of Cosine,

$$(IH)^2 = (AH)^2 + (AI)^2 - 2(AH)(AI)\cos(\frac{\beta-\theta}{2})$$

$$= 4R^2\cos^2\alpha + 16R^2\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2}) - (2)(2)R\cos\alpha(4)R\sin(\frac{\beta}{2})\sin(\frac{\theta}{2})\cos(\frac{\beta-\theta}{2})$$

$$= 4R^2\cos^2\alpha + 16R^2\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2})$$
$$-16R^2\cos\alpha\sin(\frac{\beta}{2})\sin(\frac{\theta}{2})\left[\cos(\frac{\beta}{2})\cos(\frac{\theta}{2}) + \sin(\frac{\beta}{2})\sin(\frac{\theta}{2})\right]$$

$$= 4R^2\left[\cos^2\alpha + 4\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2})\right]$$

$$-4R^2\left[4\cos\alpha\sin(\frac{\beta}{2})\sin(\frac{\theta}{2})\left(\cos(\frac{\beta}{2})\cos(\frac{\theta}{2}) + \sin(\frac{\beta}{2})\sin(\frac{\theta}{2})\right)\right]$$

$$= 4R^2\left[\cos^2\alpha + 4\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2})\right]$$

$$-4R^2\left[\cos\alpha(2)\sin(\frac{\beta}{2})\cos(\frac{\beta}{2})(2)\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}) - 4\cos\alpha\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2})\right]$$

$$= 4R^2\left[\cos^2\alpha + 8\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2})\frac{1}{2}(1-\cos\alpha) - \cos\alpha\sin\beta\sin\theta\right]$$

$$= 4R^2\left[\cos^2\alpha + 8\sin^2(\frac{\alpha}{2})\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2}) - \cos\alpha\sin\beta\sin\phi\right]$$

$$= 4R^2\left[8\sin^2(\frac{\alpha}{2})\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2}) + \cos\alpha(\cos\alpha - \sin\beta\sin\theta)\right]$$

$$= 4R^2\left[8\sin^2(\frac{\alpha}{2})\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2}) + \cos\alpha(\cos(180^\circ - (\beta+\theta)) - \sin\beta\sin\theta)\right]$$

$$= 4R^2\left[8\sin^2(\frac{\alpha}{2})\sin^2(\frac{\beta}{2})\sin^2(\frac{\theta}{2}) + \cos\alpha\sin\beta\sin\theta\right]$$

$$= 2\left[4R\sin(\frac{\alpha}{2})\sin(\frac{\beta}{2})\sin(\frac{\theta}{2})\right]^2 - 4R^2\cos\alpha\cos\beta\cos\theta$$

$$= 2r^2 - 4R^2\cos\alpha\cos\beta\cos\theta$$

$$(\mathrm{IH})^2 = 2r^2 - 4R^2\cos\alpha\cos\beta\cos\theta \qquad \text{------------ (7)}$$

then, the distance from the incenter to the orthocenter of Archimedes' the Triangle is in the form $(\mathrm{IH})^2 = 2r^2 - 4R^2\cos\alpha\cos\beta\cos\theta$

4.3 Distance from Circumcenter to Orthocenter of Archimedes' Triangle

Determine point O and H is the the circumcenter and the orthocenter of Archimedes' Triangle ABC respectively, AO is radius of the Circumscribed Circle, OH is the distance from the circumcenter to the orthocenter of the Archimedes' Triangle.

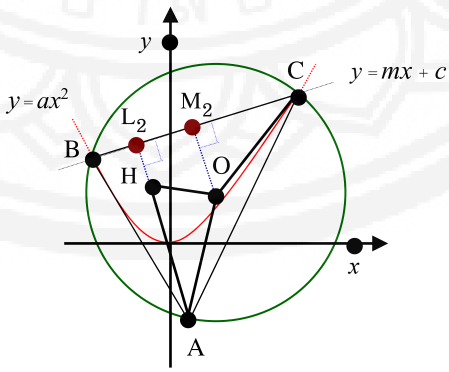


Figure 10 the circumcenter and orthocenter of Archimedes' Triangle

Determine point O and H is the circumcenter and the orthocenter of the Archimedes' triangle ABC respectively, AO is the radius of the Circumscribed circle, OH is the distance from the circumcenter to the orthocenter of the Triangle.

Let R is radius of the Circumscribed circle of Archimedes' triangle

r is radius of the Inscribed circle of Archimedes' triangle

α, β and θ is $\angle BAC$, $\angle ABC$ and $\angle ACB$ respectively

From $\angle HAC = 90^\circ - \theta$ and $\angle OAC = (90^\circ - \beta)$

thus, $\angle HAO = (90^\circ - \theta) - (90^\circ - \beta) = \beta - \theta$

Consider triangle AHO, from Law of Cosine,

$$\cos(\beta - \theta) = \frac{(AH)^2 + (AO)^2 - (OH)^2}{2(AH)(AO)}$$

$$= \frac{4R^2\cos^2\alpha + R^2 - (OH)^2}{2(2R\cos\alpha)R}$$

$$= \frac{4R^2\cos^2\alpha + R^2 - (OH)^2}{4R^2\cos\alpha}$$

$$(4R^2\cos\alpha)[\cos(\beta-\theta)] = 4R^2\cos^2\alpha + R^2 - (OH)^2$$

$$(OH)^2 = 4R^2\cos^2\alpha + R^2 - 4R^2\cos\alpha\cos(\beta-\theta)$$

$$= R^2 + 4R^2\cos\alpha\,[\cos\alpha - \cos(\beta-\theta)]$$

$$= R^2 - 4R^2\cos\alpha\,[\cos\alpha\,(\beta+\theta) - \cos(\beta-\theta)]$$

$$= R^2 - 4R^2\cos\alpha\,(2\cos\beta\cos\theta)$$

$$= R^2 - 8R^2\cos\alpha\cos\beta\cos\theta \qquad \text{------------ (8)}$$

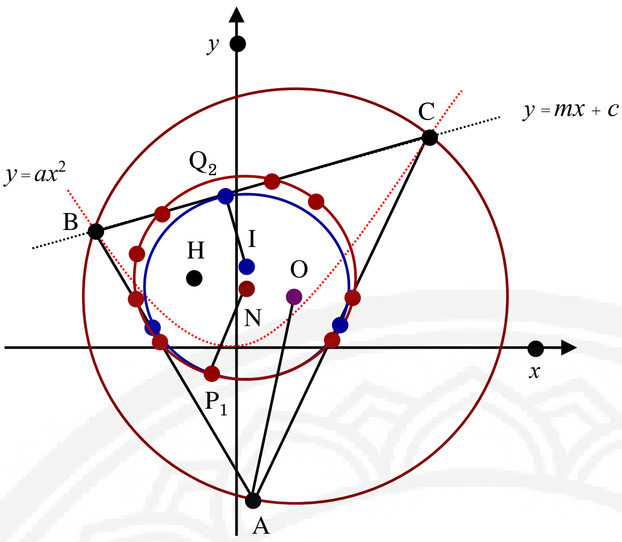
then, the distance from the circumcenter to the orthocenter of the Archimedes' Triangle is

in the form $(OH)^2 = R^2 - 8R^2\cos\alpha\cos\beta\cos\theta$

4.4 Distance from the Center of the Nine-Point Circle to the Incenter of the Archimedes' Triangle

Definition 7 Two circles are touch internally; iff the difference of the length of their radius is equal to distance between their centers.

Determine point O, I, N and H is the circumcenter, incenter, and center of the Nine-Point Circle and the orthocenter of the Archimedes' Triangle ABC respectively.



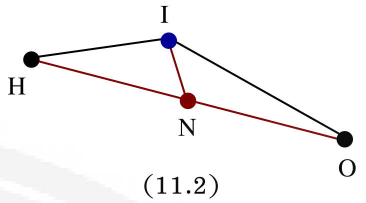


Figure 11 the Nine-Point circle and the Inscribed Circle of Archimedes' Triangle

Let AO = R is radius of the Circumscribed Circle of Archimedes' Triangle

$NP_1 = \dfrac{R}{2}$ is radius of the Nine-Point Circle of Archimedes' Triangle

$IQ_2 = r$ is radius of the Inscribed Circle of Archimedes' Triangle

IN is distance from center of the Nine-Point Circle to incenter of Archimedes' Triangle

Consider figure 11.2; because of the center of the Nine-Point Circle, circumcenter and orthocenter of the the triangle are collinear, and center of the Nine-Point Circle is the midpoint of the circumcenter to the orthocenter, (Yiu, 1998), therefore, the line is the median of triangle HIO

Determine the length of line HO, IO and HI is equal a, b and c respectively, so that

$$|\text{IN}| = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$$

$$(\text{IN})^2 = \frac{(\text{OI})^2}{2}+\frac{(\text{IH})^2}{2}-\frac{(\text{OH})^2}{4}$$

From (6), (7) and (8)

$$\begin{aligned}(\text{IN})^2 &= \frac{R^2-2Rr}{2}+\frac{2r^2-4R^2\cos\alpha\cos\beta\cos\theta}{2}-\left(\frac{R^2-8R^2\cos\alpha\cos\beta\cos\theta}{4}\right)\\ &= \frac{2R^2-4Rr+4r^2-8R^2\cos\alpha\cos\beta\cos\theta-R^2+8R^2\cos\alpha\cos\beta\cos\theta}{4}\\ &= \frac{R^2-4Rr+4r^2}{4}\\ &= \frac{(R-2r)(R-2r)}{4}\\ &= \left(\frac{R-2r}{2}\right)^2\\ \text{IN} &= \frac{R}{2}-r\end{aligned}$$

Then. the Distance from Center of the Nine-Point Circle to the Incenter of the Archimedes' Triangle is equal to the difference of the length of their radius.

5. The Relationship between the Nine-Point Circle and the Inscribed Circle of Archimedes' Triangle

From topic 4.4; the difference of the length of the radius of the Nine-Point Circle and the Inscribed Circle is equal to the distance between their centers, by definition 7 the result is two circles are touch internally.

Then, the relationship of the two circles is the Nine-Point Circle and the Inscribed Circle of Archimedes' Triangle are touch internally.

Remark

This research focus on the case parabolic curve $y = ax^2$ only, because Even though the curve place on another where in rectangular co-ordinate system, we can certainly format by sliding the axis, adjust or changing the variable to the form $y = ax^2$.

Discussion

This research point out studying about the relationship of the Nine-Point Circle, the Circumscribed Circle and the Inscribed Circle of Archimedes' Triangle by using analytical geometry and trigonometry proof. the result revealed that; the radius of the Nine-Point Circle of an Archimedes' Triangle is half of the radius of the Circumscribed Circle of the Archimedes' Triangle; this knowledge discovery is in accord with the research of Cook (1929), Court (1980) and also Hung (2011), their results showed that; the radius of the Nine-Point Circle is half of radius of the Circumscribed Circle of triangle. Other result of this research is the Nine-Point Circle and the Inscribed Circle of Archimedes' Triangle are touch internally; knowledge discovery is in accord with the research result by using projective geometry proof researches of Krishna (2016), research result by vertex changing to the form complex number of Yiu (1998) and also research result of Dekov (2009) which found that; the Nine-Point Circle and the Inscribed Circle of Archimedes' Triangle are touch internally, thus, this research is in line with academicians and according to the hypothesis.

Conclusion and Suggestion

The result of this research showed that; the radius of the Nine-Point Circle of an Archimedes' Triangle is half of the radius of the Circumscribed Circle of Archimedes' Triangle and is the Nine-Point Circle and the Inscribed Circle of Archimedes' Triangle are touch internally, this conclusion can conduce towards knowledge extending other topic about Archimedes' Triangle and the center, radius or area of the Nine-Point Circle of Archimedes' Triangle. In addition; anyone interested ought to study the property or relationship of Spieker Circle, Pedal Circle, Nagel-Point or Gergonne-Point, etc.

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