A CONVERGENCE TO INFINITY IN BANACH LATTICES
การลู่เข้าสู่ค่าอนันต์ในแลตติซบานาค

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ABSTRACT

The goal of this study is to generalize the concept of a convergence to infinity in the real field R to Banach lattices. Many possible definitions of a convergence to infinity in Banach lattices were presented. Moreover, a natural criterion was defined to determine the best possible definition for generalization.

บทคัดย่อ

วัตถุประสงค์ของงานวิจัยนี้คือการขยายแนวคิดของการลู่เข้าสู่ค่าอนันต์ในสนามของจำนวนจริงไปสู่แลตติซบานาค และได้เสนอบทนิยามของการลู่เข้าสู่ค่าอนันต์ในแลตติซบานาคที่เป็นไปได้หลายบทนิยาม ต่อจากนั้นจึงได้วางเงื่อนไขเพื่อตัดสินหาบทนิยามที่เหมาะสมที่สุดสำหรับการขยายแนวคิดดังกล่าว

INTRODUCTION

Let $(E, \|\ \|)$ be a Banach lattice. The real field R endowed with its absolute value and its usual ordering is an example of a Banach lattice. The subset $E_+ := \{x \in E \mid x \geqslant 0\}$ is called the positive cone of E; elements $x \in E_+$ are called positive. We denote by E' the set of all continuous linear functionals on E.

By an extended Banach lattice $\overleftarrow{E}$, we shall mean a structure obtained by adjoining to the Banach lattice E the ideal elements $+\infty$ and $-\infty$ and making the operational conventions:

$$x + (+\infty) = +\infty,\ x + (-\infty) = -\infty \quad \text{for all } x \in E;$$

$$\lambda(+\infty) = +\infty \text{ if } \lambda > 0;\ = -\infty \text{ if } \lambda < 0;\ = 0 \text{ if } \lambda = 0 \text{ where } \lambda \in R;$$

$$(+\infty) + (+\infty) = +\infty,\ (-\infty) + (-\infty) -\infty.$$

Also $-\infty < x < +\infty$ for all $x \in E$.

In this note we attempt to understand how a sequence (a_n) in E converges to $-\infty$ by presenting several possible definitions that can explain the nature of convergence to $-\infty$. We shall adopt a good definition (here Definition E) for our later use in the next paper which we shall study upper semi-continuous functions and subharmonic functions in Banach lattices.

METHODS

Convergence to $-\infty$ in Banach lattices

Let E be a Banach lattice and (a_n) a sequence in E. Some possible definitions of "$a_n \to -\infty$ as $n \to +\infty$" are the followings:

Definition A. The sequence (a_n) is not minorized, i.e., there is no $a_0 \in E$ such that $a_0 \leqslant a_n$ for all n.

Definition B. $(\forall\ n \in N\ [a_n \leqslant 0]\ \&\ (\| a_n \| \to +\infty$ as $n \to +\infty)$ where N denotes the set of natural number.

Definition C. $(\exists\ M \in R_+, \forall\ n \in N, [\| a_n^+ \| \leqslant M])\ \&\ (\| a_n^- \| \to +\infty$ as $n \to +\infty)$.

Definition D. $\forall\ p \in E_+, \exists\ n_0 \in N, \forall\ n \geqslant n_0\ [a_n \leqslant -p]$.

Definition E. $\exists\ M \in R_+, \forall\ p \in E_+, \exists\ n_0 \in N, \forall\ n \geqslant n_0, \exists\ q \in E$ $[a_n \leqslant -p + q$, where $\| q \| \leqslant M]$.

Let (a_n) and (b_n) be two sequences in E. We need a definition of convergence to $-\infty$ such that we can prove the following properties.

Property 1. $(a_n \to -\infty$ as $n \to +\infty)\ \&\ (b_n \to -\infty$ as $n \to +\infty) \Rightarrow$
$(a_n \vee b_n \to -\infty$ as $n \to +\infty)\ \&\ (a_n \wedge b_n \to -\infty$ as $n \to +\infty)$.

Property 2. $(a_n \to -\infty$ as $n \to +\infty)\ \&\ (\exists\ M \in R_+, \forall\ n \in N\ [\| b_n^+ \| \leqslant M]) \Rightarrow$
$(a_n + b_n \to -\infty$ as $n \to +\infty)$.

Property 3. $(a_n \to -\infty$ as $n \to +\infty) \Rightarrow \forall\ e' \in E'_+ \setminus \{0\}\ [e'(a_n) \to -\infty$ as $n \to +\infty]$.

By studying the above definitions and the given three properties together, we get the following results.

Theorem 1. For Definition A, we have

(1.1) Property 1 is true only in the case of minimum.

(1.2) Property 2 is not true.

(1.3) Property 3 is not true.

Proof. To prove (1.1), let (a_n) and (b_n) be two sequences in E. Since $a_n \wedge b_n \leqslant a_n$, then $(a_n \wedge b_n)$ is not minorized if (a_n) is not minorized. The case of maximum is not true by considering the following examples.

The space C[-1,1] of continuous real functions on [-1,1], endowed with its canonical order defined by "$f \leqslant g$ if and only if $f(t) \leqslant g(t)$ for all $t \in [-1,1]$" and the supremum norm, is a Banach lattice.

Define two sequences (a_n) and (b_n) in C[-1,1] as follows:

$$a_n(t) = \begin{cases} 0 & (-1 \leqslant t \leqslant 0), \\ -nt & (0 < t \leqslant 1), \end{cases} \qquad b_n(t) = \begin{cases} nt & (-1 \leqslant t < 0), \\ 0 & (0 \leqslant t \leqslant 1), \end{cases}$$

where n = 1,2,.... . It is obvious that the sequence (a_n) and (b_n) are not minorized. But $a_n \vee b_n = 0$ for all n, then $(a_n \vee b_n)$ is minorized.

For (1.2), consider the following example: The space $L^1(0,2)$ of Lebesgue integrable real functions on (0,2), endowed with its caconical order defined by "$f(t) \leqslant g(t)$ if and only if $f(t) \leqslant g(t)$ a.e. on (0,2)" and L^1 norm, is a Banach lattice. Define two sequences (a_n) and (b_n) in $L^1(0,2)$ as follows:

$$a_1(t) \equiv 1, \qquad b_1(t) \equiv -1,$$

$$a_n(t) = \begin{cases} n & (0 < t < 1/(n-1)), \\ -n & (2 - (1/(n-1)) < t < 2), \\ 0 & (\text{otherwise}), \end{cases}$$

and

$$b_n(t) = \begin{cases} -n & (0 < t < 1/(n-1)), \\ n & (2 - (1/(n-1)) < t < 2), \\ 0 & (\text{otherwise}), \end{cases}$$

where n = 2,3,.... . We note that the sequence (a_n) is not minorized. For, suppose not, we can find $g \in L^1(0,2)$ such that $g \leqslant a_n$ for all n. Thus, by definition of a_n, $g(t) < 1/(t-2)$ for all $t \in (1,2)$ and then $|g(t)| \geqslant |1/(t-2)|$ for $t \in (1,2)$. Hence

$$\int_0^2 |g(t)|dt \geqslant \int_1^2 |1/(t-2)|dt = +\infty.$$

This contradicts to the fact that $g \in L^1(0,2)$. So, we have the required result. Moreover, for each $n \in N$, we have

$$\|b_n^+\|_{L^1} \leqslant \int_0^2 |b_n(t)|\,dt \quad = \begin{cases} 2 & (n = 1), \\ 2n/(n-1) & (n > 2). \end{cases}$$

Hence $\| b_n^+ \|_{L^1} \leqslant 4$ for all n. But $a_n + b_n = 0$ for all n, so $(a_n + b_n)$ is minorized.

For (1.3), consider the following example, let E = C[-1,1] as in (1.1). Define a sequence (a_n) in E as follows:

$$a_n(t) \quad = \quad nt \qquad (t \in [-1,1],\ n = 1,2,.....).$$

It is obvious that (a_n) is not minorized. By choosing $e' \in E'_+ \setminus \{0\}$ to be the evaluation map at zero, i.e., $e'(f) = f(0)$, we get $e'(a_n) = 0$ for all n. Hence $e'(a_n) \not\to -\infty$ as $n \to +\infty$. This proves Theorem 1.

Theorem 2. For Definition B, we have

(2.1) Property 1 is true only in the case of minimum.

(2.2) Property 2 is not true.

(2.3) Property 3 is not true.

Proof. To prove (2.1), let (a_n) and (b_n) be two sequences in E which converges to $-\infty$ according to Definition B. For each $n \in N$, we have $a_n \wedge b_n \leqslant a_n$. Thus, by assumption, we have

$$-(a_n \wedge b_n) \geqslant -a_n \geqslant 0.$$

Hence, for each n,

$$a_n \wedge b_n \leqslant 0 \text{ and } \| a_n \wedge b_n \| \geqslant \| a_n \|.$$

This implies $a_n \wedge b_n \to -\infty$ as $n \to +\infty$ provided that $a_n \to -\infty$ as $n \to +\infty$. The case of miximum is not true. Consider the Banach lattice E = C[-1,1] and the sequences (a_n) and (b_n) as defined in (1.1). It is easily seen that $a_n \to -\infty$ and $b_n \to -\infty$ as $n \to +\infty$. But $a_n \vee b_n = 0$ for all n, so $a_n \vee b_n \not\to -\infty$ as $n \to +\infty$.

For (2.2), consider the Banach lattice E = C[0,1]. Define two sequences (a_n) and (b_n) in E as follows:

$$a_n(t) \;=\; -nt \qquad (t \in [0,1],\ n \;=\; 1,2,....),$$
$$b_n(t) \;=\; -\frac{1}{n}(t-1) \quad (t \in [0,1],\ n \;=\; 1,2,....).$$

We find that $\|b_n\|_{sup} \leqslant 1$ for all n and $a_n \to -\infty$ as $n \to +\infty$. But $a_n + b_n$ is not comparable with 0 for each n, so $a_n + b_n \not\to -\infty$.

For (2.3), consider the Banach lattice E = C[0,1] and define (a_n) as in (2.2). Choose the evaluation map $e' \in E'_+ \setminus \{0\}$ such that $e'(f) = f(0)$ where $f \in E$. We get $e'(a_n) = 0$ for all n. So $e'(a_n) \not\to -\infty$ as $n \to +\infty$.

Theorem 3. For Definition C, we have

(3.1) Property 1 is true only in the case of minimum.

(3.2) Property 2 is true.

(3.3) Property 3 is not true.

Proof. To prove (3.1), let (a_n) and (b_n) be two sequences in E which converge to $-\infty$ according to Definition C. Then there exists $M \in R_+$ such that, for each $n \in N$,

$$\| a_n^+ \| \leqslant M \text{ and } \| a_n^- \| \to +\infty \text{ as } n \to +\infty.$$

Since $a_n \wedge b_n \leqslant a_n$, Then $(a_n \wedge b_n)^+ \leqslant a_n^+$ and $(a_n \wedge b_n)^- \geqslant a_n^- \geqslant 0$.

Hence $\| (a_n \wedge b_n)^+ \| \leqslant M$ and $\| (a_n \wedge b_n)^- \| \geqslant \|a_n^-\|$ for all $n \in N$.

This proves that $a_n \wedge b_n \to -\infty$ as $n \to +\infty$. The case of maximum is not true. Consider the Banach lattice $E = C[-1,1]$ and the sequences (a_n) and (b_n) as defined in (1.1). It is easily seen that $a_n \to -\infty$ and $b_n \to -\infty$ as $n \to +\infty$. But $a_n \vee b_n = 0$ for all n, so $a_n \vee b_n \not\to -\infty$ as $n \to +\infty$.

For (3.2), let (a_n) and (b_n) be two sequences in E. Suppose that $a_n \to -\infty$ as $n \to +\infty$ and there exists $M_2 \in R_+$ such that, for each n, $\| b_n^+ \| \leqslant M_2$. Then, by Definition C, there exists $M_1 \in R_+$ such that $\| a_n^+ \| \leqslant M_1$ for all n. We observe that, for each n,

$$\begin{aligned} \| |a_n + b_n| \| &= \|(a_n + b_n)^+ + (a_n + b_n)^- \| \\ &\leqslant \|a_n^+\| + \|b_n^+\| + \|(a_n + b_n)^- \| \\ &\leqslant M_1 + M_2 + \|(a_n + b_n)^- \|, \end{aligned}$$

and

$$\begin{aligned} \| |a_n + b_n| \| &= \| a_n + b_n \| \\ &= \|a_n^+ + b_n^+) - (a_n^- + b_n^-)\| \\ &\geqslant | \| a_n^+ + b_n^+ \| - \| a_n^- + b_n^- \| | \end{aligned}$$

Hence

$$| \| a_n^+ + b_n^+ \| - \|a_n^- + b_n^-\| \leqslant M_1 + M_2 + \|(a_n + b_n)^- \|$$

Since $\|a_n^+ + b_n^+\| \leqslant M_1 + M_2$ for all n and $\|a_n^- + b_n^-\| \geqslant \|a_n^-\| \to +\infty$ as $n \to +\infty$, then $\|(a_n + b_n)^- \| \to +\infty$ as $n \to +\infty$. This proves that $a_n + b_n \to -\infty$ as $n \to +\infty$.

For (3.3), consider $E = C[-1,1]$ and define (a_n) as in (1.1), we see that $a_n \to -\infty$ as $n \to +\infty$ and $e'(a_n) = 0$ for all n where e' is the evaluation map at $x_0 = 0$.

Theorem 4. For Definition D, we have

(4.1) Property 1 is true.

(4.2) Property 2 is true in a Banach lattice which has an additional structure, i.e., in a Banach lattice where every norm bounded set is order bounded.

(4.3) Property 3 is true.

Proof. (4.1) is obvious from Definition D.

For (4.2), let (a_n) and (b_n) be two sequences in E with $a_n \to -\infty$ as $n \to +\infty$ and there exists $M \in R_+$ such that, for each n, $\| b_n^+ \| \leqslant M$. By assumption, we can find $q \in E_+$ such that

(1) $$b_n \leqslant b_n^+ \leqslant q \qquad \text{(for all n)}.$$

Let $p \in E_+$, we can find $n_0 \in N$ such that

(2) $$a_n \leqslant -p - q \qquad (n \geqslant n_0).$$

For (4.3), let (a_n) be a sequence in E with $a_n \to -\infty$ as $n \to +\infty$ and let $e' \in E'_+ \setminus \{0\}$. Let c be a positive real number. Since $e' \neq 0$, there exists $x \in E$ such that $e'(x) \neq 0$. So $e'(x^+) - e'(x^-) \neq 0$. This implies that $e'(x^+)$ or $e'(x^-) \neq 0$. Thus there exists $y \in E_+$ such that $e'(y) > 0$. Let $n_0 \in N$ be so large that $e'(n_0 y) > c$. Hence, by Definition D, there exists $n_1 \in N$ such that

$$a_n \leqslant -n_0 y \qquad (n \geqslant n_1).$$

We note that $a_n \leqslant -n_0 y$ is equivalent to $-a_n \geqslant n_0 y > 0$. Thus

$$e'(-a_n) \geqslant e'(n_0 y) > c.$$

This proves that $e'(a_n) < -c$ for all $n \geqslant n_1$. Hence $e'(a_n) \to -\infty$ as $n \to +\infty$. This completes the proof of Theorem 4.

We observe that it is difficult to find a counter example for (4.2) since we can hardly find examples for $a_n \to -\infty$ according to Definition D. However, there are such examples for Definition E. In fact, $-ne \to -\infty$ for almost units e of E.

Theorem 5. For Definition E, we have

(5.1) Property 1 is true.

(5.2) Property 2 is true.

(5.3) Property 3 is true.

Proof. To prove (5.1), let (a_n) and (b_n) be two sequences in E such that $a_n \to -\infty$ and $b_n \to -\infty$ as $n \to +\infty$. Then there exists $M \in R_+$ such that for each $p \in E_+$ we can find $n_1, n_2 \in N$ such that

(3) $\forall\, n \geq n_1,\ \exists q_1 \in E\ [a_n \leq -p + q_1$, where $\| q_1 \| \leq M]$, and

(4) $\forall\, n \geq n_2,\ \exists q_2 \in E\ [a_n \leq -p + q_2$, where $\| q_2 \| \leq M]$.

Since, for each n, $a_n \wedge b_n \leq a_n$, it follows from (3) that $a_n \wedge b_n \to -\infty$ as $n \to +\infty$. For the case of maximum: We observe that, for each $n \geq \max \{n_1, n_2\}$, we have

$$a_n \leq -p + q_1 \leq -p + |q_1|,$$

and

$$b_n \leq -p + q_2 \leq -p + |q_2|.$$

Hence

(5) $$a_n \wedge b_n \leq -p + |q_1| + |q_2|,$$

where $|q_1| + |q_2| \in E$ and $\| \, |q_1| + |q_2| \, \| \leq M + M = 2M$. Therefore (5) implies $a_n \wedge b_n \to -\infty$ as in $\to +\infty$.

For (5.2), let (a_n) and (b_n) be two sequences in E such that $a_n \to -\infty$ as $n \to +\infty$ and there exists $M_1 \in R_+$ such that $\|b_n^+\| \leq M_1$ for all n. Then there exists $M_2 \in R_+$ such that for each $p \in E_+$ we can find $n_0 \in N$ such that

(6) $\forall n \geq n_0,\ \exists q \in E\ [a_n \leq -p + q$, where $\|q\| \leq M_2]$.

Thus, for each $n \geq n_0$, we have

$$a_n + b_n \leq -p + b_n^+ + q,$$

where $b_n^+ + q \in E$ and $\| b_n^+ + q \| \leq M_1 + M_2$. This proves that $a_n + b_n \to -\infty$ as $n \to +\infty$.

For (5.3), let $e' \in E'_+ \setminus \{0\}$ and assume that $a_n \to -\infty$ as $n \to +\infty$. Then there exists $M \in R_+$ such that for each $p \in E_+$ we can find $n_0 \in N$ such that

(7) $\qquad \forall\, n \geq n_0,\ \exists\, q \in E\ [a_n \leq -p + q, \text{ where } \|q\| \leq M].$

Since e' is continuous, the set $\{e'(x) \mid x \in E,\ \|x\| \leq M\}$ is bounded by M_1, say. Let $y \in E_+$ be such that $e'(y) > 0$ (such a y exists since $e' \neq 0$). Let $c \in R_+$, choose $n_1 \in N$ be so large that

$$e'(n_1 y) - M_1 > c.$$

Replace p in (7) by $n_1 y$, we can find $n_2 \in N$ such that

$$\forall\, n \geq n_2,\ \exists\, r \in E\ [a_n \leq -n_1 y + r, \text{ where } \|r\| \leq M].$$

Hence, for all $n \geq n_2$, we get

$$-a_n - (n_1 y - r) \geq 0.$$

Thus

$$-e'(a_n) \geq e'(n_1 y - r),$$

and then

$$e'(a_n) \leq -e'(n_1 y) + e'(r) \leq -e'(n_1 y) + M_1 \leq -c.$$

This proves that $e'(a_n) \to -\infty$ as $n \to +\infty$.

Remark

1. It follows from Theorem 5 that Definition E has all necessary properties that we should have for a sequence which converges to $-\infty$. So we shall adopt Definition E for our later use.

2. Consider the Banach lattice $E = L^1(0,2)$ as in the proof of (1.2). Define a sequence (a_n) in E as follows:

$$a_n(t) = -ne \qquad (t \in (0,2),\ n = 1,2,\ldots),$$

where $e \in R_+ \setminus \{0\}$. We find that $a_n \not\to -\infty$ as $n \to +\infty$ according to Definition D as, for each n, a_n is not comparable with the negative element $-p$ where $p(t) = 1/\sqrt{t}$. But we still have $a_n \to -\infty$ as $n \to +\infty$ according to Definition E. To prove this, let $p \in L^1(0,2)_+$ and choose $M = 1$. For each $n \in N$, we define

$$A_n = \{t \in (0,2) \mid p(t) \geq ne\}.$$

Since $p \in L^1(0,2)$, then $m(A_n) \to 0$ as $n \to +\infty$ where m is the Lebesgue measure on the real interval (0,2). So we can find $n_0 \in N$ such that

$$\int_{A_{n_0}} p(t)dm(t) \leq 1.$$

Define a function q on (0,2) by

$$q(t) = \begin{cases} 0 & (t \in A_{n_0}), \\ p(t) & (t \in A_{n_0}). \end{cases}$$

Thus $q \in L^1(0,2)$ and we have

$$-ne = a_n \leq -n_0 e \leq -p(t) + q(t) \qquad (n \geq n_0),$$

where $\|q\| L^1 \leq 1$. This implies $a_n \to -\infty$ as $n \to +\infty$.

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