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Full Paper

Symmetries and exact solutions of Einstein field equations for perfect fluid distribution and pure radiation fields

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Abstract: Lie group formalism is applied to Einstein field equations for perfect fluid distribution and pure radiation fields in the investigation of symmetries and exact solutions. The similarity reductions are obtained by determining the complete sets of point symmetries of these equations. The reduced ordinary differential equations are further studied and some non-trivial exact solutions are successfully furnished.

Keywords: Lie classical method, Einstein field equations, perfect fluid distribution, pure radiation fields

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INTRODUCTION

It is well known that complex physical phenomena are related to non-linear partial differential equations, which are involved in many fields of science, especially in optical fibres, chemical kinematics, chemical physics and general relativity. Further, the investigation of exact solutions of non-linear partial differential equations has an important role in the study of non-linear physical phenomena. The literature abounds with many different techniques that have been invoked in an effort to obtain new exact solutions for different configurations of matter.

Lie group analysis method [1-3], also called the symmetry method, is one of the most effective methods for determining solutions of non-linear partial differential equations. Since the second half of the 19th century and about 200 years after Leibniz and Newton introduced the concept of the derivative, solving ordinary differential equations (ODEs) has been one of the most important problems in applied mathematics. Sophus Lie (1842-1899) became interested in this problem and with inspiration from Galois's theory [1] for solving algebraic equations discovered what is known today as Lie group analysis. Lie showed that the majority of known methods of integration of ordinary differential equations, which until then had seemed artificial, could be derived in a unified manner using his theory of continuous transformation groups [3]. Recently there have been considerable developments in finding exact solutions of non-linear differential equations, as evident by a number of research work [4-6]. For many years after Einstein proposed his general theory of relativity, only a few exact solutions were known. Today the situation is completely different and we now have a vast number of solutions of Einstein field equations for various fields [7]. However, very few are well understood in the sense that they can be clearly interpreted as the fields of real physical sources. The obvious exceptions are the Schwarzschild [8] and Kerr solutions [9], which have been very thoroughly analysed and which clearly describe the gravitational fields surrounding static and rotating black holes respectively.

Thus, the study of exact solutions to Einstein field equations for various fields is an important part of the theory of general relativity. Einstein field equations, which play a central role in Einstein theory of general relativity, have symmetry consideration as one of the most important mathematical properties apart from applications and implications in astrophysics. The heart of the classification schemes for the solutions of these equations is the symmetry methods based on the Lie group. Einstein field equations were studied by various authors [6, 10, 11, 12] to establish exact solutions by using Lie group analysis.

In this paper, we study the exact solutions of Einstein field equations for perfect fluid distribution and pure radiation fields. Lie symmetry method is used to generate various symmetries of the equations and then an optimal system comprising basic vector fields is identified, and finally the reduced systems of ODEs and their exact solutions are presented. The exact solutions thus obtained can be utilised for checking the validity of numerical and approximation techniques and programmes of the theory of general relativity.

EINSTEIN FIELD EQUATIONS FOR PERFECT FLUID DISTRIBUTION

The possibility of the existence of gravitational waves propagated with the speed of light was first pointed out by Einstein in the case of weak gravitational field [13]. The usual procedure in Cartesian coordinates is to start with a field:

$$g_{ik} = \eta_{ik} + h_{ik} \qquad i,k = 1,2,3,4, \tag{1}$$

where η_{ik} is the Galilean metric and h_{ik} describes the modifications due to a weak gravitational field. In view of the linearised field equations $R_{ik} = 0$ coupled with a set of coordinates conditions, h_{ik} satisfies the wave equation. In particular, when h_{ik} depends on t and x only, there exists a coordinate system [14] in which one can take all the components h_{ik} to vanish except

$$h_{22} = -h_{33} \neq 0, \quad h_{23} = -h_{32} \neq 0 \tag{2}$$

where the non-vanishing components are arbitrary functions of the argument $(t-x)$. Since general relativity is essentially a non-linear theory, its intrinsic consequences cannot be based on a weak field approximation and there must be certain reservations about the conclusions drawn from the linearised field. Bondi et al. [15] demonstrated the existence of plane gravitational waves described by an exact solution of Einstein field equations for empty space-time. In the present paper, we consider the exact gravitational field equations:

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$$R_{ik} = -8\pi\left(T_{ik} - \frac{1}{2}g_{ik}T\right) \qquad (3)$$

for a line element:

$$ds^2 = dt^2 - dx^2 - (1-u)dy^2 - (1+u)dz^2 + 2vdydz \qquad (4)$$

where u and v are functions of t and x only.

In the case of the line element (4), the non-zero components of the curvature tensor and the Ricci tensor are given as follows:

$$R_{yzyz} = \frac{u_x^2 - u_t^2 + v_x^2 - v_t^2}{4}$$

$$R_{z\mu z\nu} = \frac{2Pu_{\mu\nu} - (1-u)u_\mu u_\nu - (1+u)v_\mu v_\nu + v(u_\mu v_\nu + v_\mu u_\nu)}{4P}$$

$$R_{y\mu y\nu} = \frac{-(2Pu_{\mu\nu} + (1+u)u_\mu u_\nu + (1-u)v_\mu v_\nu + v(u_\mu v_\nu + v_\mu u_\nu))}{4P}$$

$$R_{y\mu z\nu} = \frac{-(2Pu_{\mu\nu} - (1-u)u_\mu u_\nu + (1+u)v_\mu v_\nu - v(u_\mu v_\nu - v_\mu u_\nu))}{4P}$$

$$R_{yztx} = \frac{(u_t v_x - v_t u_x)}{2P}$$

$$R_{\mu\nu} = \frac{-(2P(uu_{\mu\nu} + vv_{\mu\nu}) + (1+u^2-v^2)u_\mu u_\nu + (1-u^2+v^2)v_\mu v_\nu + 2uv(u_\mu v_\nu + v_\mu u_\nu))}{2P^2}$$

$$R_{yy} + R_{zz} = \frac{u_x^2 - u_t^2 + v_x^2 - v_t^2}{P}$$

$$R_{yy} - R_{zz} = \frac{P(u_{xx} - u_{tt}) - u(v_x^2 - v_t^2) + v(u_x v_x - v_t u_t)}{P}$$

$$R_{yz} = R_{zy} = \frac{P(v_{xx} - v_{tt}) - v(u_x^2 - u_t^2) + u(u_x v_x - v_t u_t)}{P} \qquad (5)$$

where μ and ν take the values t and x only, and $u_i \equiv \frac{\partial u}{\partial x^i}$, $u_{ik} \equiv \frac{\partial u}{\partial x^i \partial x^k}$,... etc., and $\left(x^1, x^2, x^3, x^4\right) = (t, y, z, x)$ and $P = (1 - u^2 - v^2)$.

The Perfect Fluid Distribution

We examine the compatibility of the perfect fluid distribution of matter defined by the field equations:

$$R_{ik} = -8\pi[(p+\rho)v_i v_k - \frac{1}{2}g_{ik}(\rho - p)],\ g^{ik}v_i v_k = 1, \qquad (6)$$

where p and ρ are the proper pressure and proper density respectively and v_i is the flow vector. In view of (5) and (6), we have the following four relations:

$$(1-u)^{-1}R_{yy} = (1+u)^{-1}R_{zz} = -v^{-1}R_{yz}$$
$$((1-u)R_{tt} - R_{yy})((1-u)R_{xx} + R_{yy}) = (1-u)^2 R_{tx}^2. \qquad (7)$$

Two of the relations, contained in the first set of the above equations, give:

$$P(u_{xx} - u_{tt}) + u(u_x^2 - u_t^2) + u(u_x v_x - u_t v_t) = 0$$
$$P(v_{xx} - v_{tt}) + v(v_x^2 - v_t^2) + u(u_x v_x - u_t v_t) = 0. \quad (8)$$

A perfect fluid distribution of matter is possible if $u = v$.

Thus these relations are compatible with the perfect fluid distribution of matter if $u = v$ and the resulting single equation is as follows:

$$(1 - 2u^2)(u_{tt} - u_{xx}) + 2u(u_t^2 - u_x^2) = 0, \quad (9)$$

Lie Symmetry Analysis

Lie point symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. The technique has earlier been used to obtain exact solutions of various non-linear partial differential equations [4-6, 10, 11]; hence there is no need to discuss the method in detail. In this section, we obtain the symmetry groups of equation (9) using the Lie classical method. The symmetry group of equation (9) is generated by a vector field of the form:

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \quad (10)$$

where ξ, τ and η are functions of x, t and u. Assuming that the system of equation (9) is invariant, we find that the coefficient functions ξ, τ and η must satisfy the symmetry condition:

$$-4\eta u(u_{tt} - u_{xx}) + (1 - 2u^2)(\eta^{tt} - \eta^{xx}) + 2\eta(u_t^2 - u_x^2) + 4u(u_t \eta^t - u_x \eta^x) = 0 \quad (11)$$

where η^x, η^t, η^{xx} and η^{tt} are extended (prolonged) infinitesimals acting on an enlarged space that includes derivatives of the dependent variables u_x, u_t, u_{xx} and u_{tt} respectively. After some straightforward albeit tedious and lengthy calculations, we derive the following forms of the infinitesimal elements ξ, τ and η:

$$\xi = F_1(t + x) - F_2(t - x)$$
$$\tau = F_1(t + x) + F_2(t - x) \quad (12)$$
$$\eta = 0$$

where $F_1(t + x)$ and $F_2(t - x)$ are arbitrary functions. Thus, equation (9) admits a set of Lie algebra of infinite dimensions.

For the symmetries described in (12), the similarity variable $\zeta = \zeta(x,t)$ and the corresponding form of u as a function of the new independent variable ζ are as follows:

$$\zeta = \int \frac{F_1(t + x) + F_2(t - x)}{F_1(t + x)F_2(t - x)}dx + \int \frac{-F_1(t + x) + F_2(t - x)}{F_1(t + x)F_2(t - x)}dt$$
$$u(x,t) = F(\zeta). \quad (13)$$

In the above set of equations (13), the function F is a function of ζ and is determined by substituting (13) in (9) and solving the resulting non-linear ODE, which is:

$$2F'(\zeta)^2 F(\zeta) + F''(\zeta) - 2F(\zeta)^2 F''(\zeta) = 0, \quad (14)$$

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where prime (') denotes the differentiation with respect to variable ζ . Solving equation (14) and reverting back to the original variables, we obtain the following group-invariant solutions of equation (9):

Solutions in terms of cos () function

$$(i)\ u(x,t)=\pm\frac{\sqrt{2}}{2}\cos\left(c_1+c_2\left(\int\frac{F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dx+\int\frac{-F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dt\right)\right)$$

$$(ii)\ u(x,t)=\pm\frac{\sqrt{2}}{2}\mp\sqrt{2}\cos\left(c_1+c_2\left(\int\frac{F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dx+\int\frac{-F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dt\right)\right)^2 \quad (15)$$

$$(iii)\ u(x,t)=\pm\frac{3\sqrt{2}}{2}\cos\left(c_1+c_2\left(\int\frac{F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dx+\int\frac{-F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dt\right)\right)$$
$$\mp 2\sqrt{2}\cos\left(c_1+c_2\left(\int\frac{F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dx+\int\frac{-F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dt\right)\right)^3,$$

where c_1 and c_2 are arbitrary constants.

Solutions in terms of sin () function

$$(i)\ u(x,t)=\pm\frac{\sqrt{2}}{2}\sin\left(c_1+c_2\left(\int\frac{F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dx+\int\frac{-F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dt\right)\right)$$

$$(ii)\ u(x,t)=\pm\frac{\sqrt{2}}{2}\mp\sqrt{2}\sin\left(c_1+c_2\left(\int\frac{F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dx+\int\frac{-F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dt\right)\right)^2 \quad (16)$$

$$(iii)\ u(x,t)=\pm\frac{3\sqrt{2}}{2}\sin\left(c_1+c_2\left(\int\frac{F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dx+\int\frac{-F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dt\right)\right)$$
$$\mp 2\sqrt{2}\sin\left(c_1+c_2\left(\int\frac{F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dx+\int\frac{-F_1(t+x)+F_2(t-x)}{F_1(t+x)F_2(t-x)}dt\right)\right)^3,$$

where c_1 and c_2 are arbitrary constants.

EINSTEIN FIELD EQUATIONS FOR PURE RADIATION FIELDS

From the beginning of the theory of general relativity, there has been a sustained search for the new exact solutions of Einstein equations for various fields. The exact solutions of Einstein field equations play very important roles in the discussion of physical problems. The Riemann curvature tensor plays the most fundamental role in Einstein theory of gravitation. The algebraic and differential properties of this tensor have characterised wave fields in general relativity in great detail. The problem of pure radiation fields has been discussed by several authors [16-22].

The field equations corresponding to the pure radiation fields are:

$$R_i^{\ j}=\kappa\omega_i\omega^j, \quad (17)$$

where κ is a scalar. When $\kappa=0$, one gets pure gravitational radiation. The more general waves given by (17) ($\kappa\neq 0$) are distinct from pure gravitational waves. We will derive some of the exact

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solutions of the Einstein-Rosen [23] cylindrically symmetric space-time corresponding to pure radiation fields.

Metric Form and Field Equations

Consider Einstein-Rosen metric [23] in cylindrical polar coordinates r, ϕ, z and time t as:

$$ds^2 = e^{(2v-2u)}(dt^2 - dr^2) - r^2 e^{(-2u)} d\phi^2 - e^{(2u)} dz^2 \ , \tag{18}$$

where u and v are functions of r and t only. The non-zero components of curvature tensor obtained from (18) are:

$$\begin{aligned} R_r^r &= e^{(2u-2v)}\left(-v_{rr} + v_{tt} + u_{rr} - u_{tt} - 2u_r^2 + \frac{v_r + u_r}{r}\right) \\ R_\phi^\phi &= -R_z^z = e^{(2u-2v)}\left(u_{rr} - u_{tt} + \frac{u_r}{r}\right) \\ R_t^t &= e^{(2u-2v)}\left(-v_{rr} + v_{tt} + u_{rr} - u_{tt} + 2u_t^2 + \frac{u_r - v_r}{r}\right) \\ R_r^t &= -R_t^r = e^{(2u-2v)}\left(2u_r u_t - \frac{v_t}{r}\right). \end{aligned} \tag{19}$$

Pure radiation fields with null vector ω^i such that $\omega^r = 1, \omega^\phi = 0, \omega^z = 0$ and $\omega^t = 1,$ for the metric (18) by using (17), obey the field equations:

$$\begin{aligned} R_r^r + R_t^t &= 0 \\ R_r^r + R_t^r &= 0 \\ R_\phi^\phi + R_z^z &= 0. \end{aligned} \tag{20}$$

Making use of expressions for R_i^i given in (19), the relations (20) give the system of partial differential equations:

$$\begin{aligned} &u_{rr} + \frac{u_r}{r} - u_{tt} = 0 \\ &v_r + v_t - r(u_r + u_t)^2 = 0 \\ &v_{rr} - v_{tt} + u_r^2 - u_t^2 = 0. \end{aligned} \tag{21}$$

So we have three equations for the determination of two unknowns, u and v, and one can easily verify that these three equations are all consistent. Therefore, we drop the third equation in system (21) and solve the remaining two equations for u and v. Hence we get a system of partial differential equations:

$$\begin{aligned} &u_{rr} + \frac{u_r}{r} - u_{tt} = 0 \\ &v_r + v_t - r(u_r + u_t)^2 = 0. \end{aligned} \tag{22}$$

Lie symmetry method is utilised to obtain the group invariant solutions of the non-linear system (22). A number of cases arise depending on the nature of the Lie symmetry generator. We derive various symmetries of system (22) by using Lie group method and then an optimal system

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comprising basic vector fields is identified. Further, the reduced systems of ODEs and some of the exact solutions of equation (22) are presented. The Lie algebra associated with system (22) consists of the following six vector fields:

$$X_1 = u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v},\ X_2 = \log(r)\frac{\partial}{\partial u} + 2u\frac{\partial}{\partial v},\ X_3 = \frac{\partial}{\partial v},\ X_4 = \frac{\partial}{\partial u},\ X_5 = r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t},\ X_6 = \frac{\partial}{\partial t}. \quad (23)$$

In general, there is an infinite number of sub-algebras of this Lie algebra formed from any linear combination of generators X_i, $i = 1,2,3,4,5,6$. However, two algebras are similar if they are connected to each other by a transformation from the symmetry group; then their corresponding invariant solutions are connected to each other by the same transformation. Therefore, it is sufficient to put all similar sub-algebras into one class; the set of all these representatives is called an optimal system [1, 2], which consists of the following six basic vector fields:

$$(i)\ X_1 + \mu X_5,\ \ (ii)\ X_2 + \mu X_5,\ \ (iii)\ X_3 + \mu X_5,\ \ (iv)\ X_4 + \mu X_5,\ \ (v)\ X_5,\ \ (vi)\ X_6. \quad (24)$$

Symmetry Reductions and Exact Solutions

Now the primary focus is on the reductions associated with the vector fields in the optimal system and the attempt to furnish exact solutions.

(i) $X_1 + \mu X_5$

Corresponding to this vector field, the form of the similarity variable and similarity solution is:

$$\zeta = \frac{r}{t},\ u(r,t) = t^{\frac{1}{\mu}} F(\zeta),\ v(r,t) = t^{\frac{2}{\mu}} G(\zeta).$$

On using these in system (22) , the reduced system of ODEs is:

$$\begin{aligned} &F''(\zeta)(1-\zeta^2) + \frac{2\zeta F'(\zeta)}{\mu} + \frac{F'(\zeta)}{\mu} - 2F'(\zeta)\zeta - \frac{F(\zeta)}{\mu^2} + \frac{F(\zeta)}{\mu} = 0 \\ &G''(\zeta) + \frac{2G(\zeta)}{\mu} - G'(\zeta)\zeta - \zeta\left(\frac{F(\zeta)}{\mu} - \zeta F'(\zeta) + F'(\zeta)\right)^2 = 0 \end{aligned} \quad (25)$$

The solution of the reduced system of ODEs (25) is:

$$\begin{aligned} F(\zeta) &= c_1 hypergeom\left(\left[\frac{-1}{2\mu}, \frac{\mu-1}{2\mu}\right], \left[\frac{\mu-2}{2\mu}\right], 1-\zeta^2\right) \\ &\quad + c_2(-1+\zeta^2)^{\frac{2+\mu}{2\mu}} hypergeom\left(\left[\frac{1+2\mu}{2\mu}, \frac{\mu+1}{2\mu}\right], \left[\frac{3\mu+2}{2\mu}\right], 1-\zeta^2\right) \\ G(\zeta) &= \left(\int \frac{-\zeta(F'(\zeta)\zeta\mu - F(\zeta) - F'(\zeta)\mu)^2(-1+\zeta)^{\frac{-\mu-2}{2}}}{\mu} d\zeta + c_1\right)(-1+\zeta^2)^{\frac{2}{\mu}}. \end{aligned} \quad (26)$$

Hence, the solution of system (22) is:

$$u(r,t)=\left(c_1 hypergeom\left(\left[\frac{-1}{2\mu},\frac{\mu-1}{2\mu}\right],\left[\frac{\mu-2}{2\mu}\right],\left(1-\left(\frac{r}{t}\right)^2\right)\right)\right)t^{\frac{1}{\mu}}$$
$$+\left(c_2\left(-1+\left(\frac{r}{t}\right)^2\right)^{\frac{2+\mu}{2\mu}} hypergeom\left(\left[\frac{1+2\mu}{2\mu},\frac{\mu+1}{2\mu}\right],\left[\frac{3\mu+2}{2\mu}\right],\left(1-\left(\frac{r}{t}\right)^2\right)\right)\right)t^{\frac{1}{\mu}} \tag{27}$$
$$v(r,t)=\left(\left(\int\frac{-\zeta(F'(\zeta)\zeta\mu-F(\zeta)-F'(\zeta)\mu)^2(-1+\zeta)^{\frac{-\mu-2}{2}}}{\mu}d\zeta+c_1\right)(-1+\zeta^2)^{\frac{2}{\mu}}\right)t^{\frac{2}{\mu}}.$$

where c_1 and c_2 are arbitrary constants and *hypergeom* stands for hypergeometric function.

(*ii*) $X_2+\mu X_5$

For this vector field, the form of the similarity variable and similarity solution is:

$$\zeta=\frac{r}{t},\ u(r,t)=\frac{(\log(r))^2}{2\mu}+F(\zeta),\ v(r,t)=\frac{(\log(r))^3}{3\mu^2}+\frac{2F(\zeta)\log(r)}{\mu}+G(\zeta).$$

On using these in system (22), the following system of reduced ODEs is obtained:

$$\begin{aligned}&F''(\zeta)\mu\zeta^2(1-\zeta^2)-2\mu\zeta^3F'(\zeta)+\mu F'(\zeta)\zeta+1=0\\&\mu\zeta^2G'(\zeta)-\mu\zeta G'(\zeta)+(F(\zeta))^2\zeta^4\mu-2(F'(\zeta))^2\zeta^3\mu+(F'(\zeta))^2\zeta^2\mu-2(F(\zeta))=0.\end{aligned} \tag{28}$$

The solution of system (28) is given by:

$$\begin{aligned}F(\zeta)&=\int\frac{-\arctan\left(\frac{1}{\sqrt{(-1+\zeta^2)}}\right)\sqrt{(-1+\zeta^2)}+c_2\mu\sqrt{(-1+\zeta^2)}}{(\zeta^2-1)\zeta\mu}d\zeta+c_3\\G(\zeta)&=\int\frac{-(F(\zeta))^2\zeta^4\mu+2(F'(\zeta))^2\zeta^3\mu-(F'(\zeta))^2\zeta^2\mu+2(F(\zeta))}{\mu\zeta(\zeta-1)}d\zeta+c_1.\end{aligned} \tag{29}$$

where c_1, c_2 and c_3 are arbitrary constants. Thus, we get the following solution of system (22) by using (29):

$$\begin{aligned}u(r,t)&=\frac{(\log(r))^2}{2\mu}+F(\zeta)\\v(r,t)&=\frac{(\log(r))^3}{3\mu^2}+\frac{2F(\zeta)\log(r)}{\mu}+G(\zeta).\end{aligned} \tag{30}$$

(*iii*) $X_3+\mu X_5$

For this vector field, the form of the similarity variable and similarity solution is:

$$\zeta=\frac{r}{t},\ u(r,t)=F(\zeta),\ v(r,t)=\frac{\log(t)}{\mu}+G(\zeta).$$

Using these substitutions, system (22) reduces to:

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$$F''(\zeta)(1-\zeta^2)-2\zeta F'(\zeta)+\frac{F'(\zeta)}{\zeta}=0$$
$$G'(\zeta)(1-\zeta)-\zeta(-\zeta F'(\zeta)+F'(\zeta))^2+\frac{1}{\mu}=0. \tag{31}$$

The solution of the reduced system of ODEs (31) is obtained and the solution of system (22) is:

$$u(r,t)=\arctan\left(\frac{t}{\sqrt{(r^2-t^2)}}\right)c_2+c_1$$
$$v(r,t)=-c_2^2\log(r)+c_2^2\log(r+t)+\frac{1}{\mu}\log(r-t)+c_3, \tag{32}$$

where c_1, c_2 and c_3 are arbitrary constants.

(iv) $X_4+\mu X_5$

In this case the form of the similarity variable and similarity solution is:

$\zeta=\frac{r}{t},\ u(r,t)=F(\zeta)+\frac{\log(r)}{\mu},\ v(r,t)=G(\zeta).$

Using these substitutions in system (22), we get the following reduced system of ODEs:

$$F''(\zeta)(1-\zeta^2)-2\zeta F'(\zeta)+\frac{F'(\zeta)}{\zeta}+\frac{1}{\mu}=0$$
$$G'(\zeta)(1-\zeta)-\zeta\left(-\zeta F'(\zeta)+F'(\zeta)+\frac{1}{\mu}\right)^2=0. \tag{33}$$

The solution of reduced ODEs (33) is obtained and hence the solution of system (22) is:

$$u(r,t)=\frac{\log(r)}{\mu}-c_1\arctan\left(\frac{t}{\sqrt{(r^2-t^2)}}\right)+c_2$$
$$v(r,t)=-\frac{\log(r-t)}{\mu^2}+\frac{\log(r)}{\mu^2}-c_1^2\log(r)+c_1^2\log(r+t)-\frac{2c_1\arctan\left(\frac{t}{\sqrt{(r^2-t^2)}}\right)}{\mu}+c_3, \tag{34}$$

where c_1, c_2 and c_3 are arbitrary constants.

(v) X_5

For this vector field, the form of the similarity variable and similarity solution is:

$\zeta=\frac{r}{t},\ u(r,t)=F(\zeta),\ v(r,t)=G(\zeta).$

Using these substitutions, system (22) reduces to:

$$F''(\zeta)(1-\zeta^2)-2\zeta F'(\zeta)+\frac{F'(\zeta)}{\zeta}=0$$
$$G'(\zeta)(1-\zeta)-\zeta(-\zeta F'(\zeta)+F'(\zeta))^2=0. \tag{35}$$

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The solution of reduced ODEs (35) is furnished and the solution of system (22) is:

$$\begin{aligned} u(r,t) &= c_1 \arctan\left(\frac{t}{\sqrt{(r^2-t^2)}}\right) + c_2 \\ v(r,t) &= -c_1^2 \log(r) + c_1^2 \log(r+t) + c_3, \end{aligned} \tag{36}$$

where c_1, c_2 and c_3 are arbitrary constants.

(vi) X_6

Corresponding to this vector field, the form of the similarity variable and similarity solution is:

$\zeta = r,\ u(r,t) = F(\zeta),\ v(r,t) = G(\zeta).$

On using these in system (22), the system of reduced ODEs is given by:

$$\begin{aligned} &\zeta F''(\zeta) + F'(\zeta) = 0 \\ &G'(\zeta) - \zeta (F'(\zeta))^2 = 0. \end{aligned} \tag{37}$$

The solution of reduced ODEs (37) is obtained and the solution of system (22) is deduced as:

$$\begin{aligned} u(r,t) &= c_1 + c_3 \log(r) \\ v(r,t) &= c_2 + c_3^2 \log(r), \end{aligned} \tag{38}$$

where c_1, c_2 and c_3 are arbitrary constants.

Since, after reduction to ODEs, further attempt to apply Lie group analysis to ODEs has been made, but no further physically important non-trivial symmetries come out, hence the solutions of ODEs are obtained directly. After attaining the reductions and exact solutions corresponding to essential vector fields of the optimal system, we observe that in each of physically relevant case, the similarity variable is of the form r/t . Since reductions can be obtained from any linear combination of basic vector fields (23), we can consider other linear combinations for physically significant reductions and exact solutions.

For example, we consider linear combination $X_1 + \mu X_2 + \lambda X_6$ of vector fields. For this vector field, the form of the similarity variable and similarity solution is:

$$\zeta = r,\ u(r,t) = e^{\frac{t}{\lambda}} F(\zeta) - \mu \log(\zeta),\ v(r,t) = -2\mu e^{\frac{t}{\lambda}} F(\zeta) + \mu^2 \log(\zeta) + e^{\frac{2t}{\lambda}} G(\zeta).$$

On using these in system (22), the reduced system of ODEs is:

$$\begin{aligned} &-F''(\zeta)\zeta\lambda^2 + F(\zeta)\zeta - \lambda^2 F'(\zeta) = 0 \\ &-G'(\zeta)\lambda^2 - 2G(\zeta)\lambda + F(\zeta)^2\zeta + 2F(\zeta)F'(\zeta)\zeta\lambda + F'(\zeta)^2\lambda^2\zeta = 0. \end{aligned} \tag{39}$$

The solution of reduced ODEs is obtained and the solution of system (22) is deduced as:

$$u(r,t)=\left(c_2 J_0\left(\frac{r}{\lambda}\right)+c_3 Y_0\left(\frac{r}{\lambda}\right)\right)e^{\frac{t}{\lambda}}-\mu\log(r)$$

$$v(r,t)=\left(\int re^{\left(\frac{2r}{\lambda}\right)}\left(c_2 J_1\left(\frac{r}{\lambda}\right)-c_3 Y_1\left(\frac{r}{\lambda}\right)+c_2 J_0\left(\frac{r}{\lambda}\right)-c_3 Y_0\left(\frac{r}{\lambda}\right)\right)^2 dr+c_1\lambda^2\right)\frac{e^{\left(\frac{-2r}{\lambda}\right)}e^{\left(\frac{2t}{\lambda}\right)}}{\lambda^2} \tag{40}$$

$$+\mu^2\log(r)-2\mu e^{\left(\frac{t}{\lambda}\right)}\left(c_2 J_0\left(\frac{r}{\lambda}\right)+c_3 Y_0\left(\frac{r}{\lambda}\right)\right),$$

where $J_\ell(x)$ and $Y_\ell(x)$ are the modified Bessel functions of the first and second kinds respectively. They satisfy the modified Bessel equation:

$$x^2Y''+xY'-(x^2+\ell^2)Y=0,$$

where c_1, c_2 and c_3 are arbitrary constants.

CONCLUSIONS

In this work, we have studied Einstein field equations for perfect fluid distribution and the system of partial differential equations corresponding to Einstein-Rosen cylindrically symmetric space-time for pure radiation fields by using Lie symmetry analysis method. Especially, all similarity reductions and exact solutions based on the Lie group method are obtained by generating the group infinitesimals. The partial differential equations are reduced to ordinary differential equations, which are further studied with the aim of deriving certain exact solutions. It is worth mentioning here that the authenticity of all the solutions has been checked with the aid of software Maple. Thus, we have found new exact solutions that might prove to be interesting for further applications.

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