# ไอดีลเฉพาะของเนียร์เลฟท์ออลโมซท์ริง
# Prime Ideals of Near Left Almost Rings

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**บทคัดย่อ**

ในงานวิจัยชิ้นนี้เป็นการขยายงานวิจัยของ T. Shah, F. Rehman and M. Raees ซึ่งเราได้นิยามและศึกษาคุณสมบัติของไอดีลเฉพาะ และไอดีลใหญ่สุดของเอ็นแอลเอริง ถ้า *M* เป็นไอดีลใหญ่สุดเฉพาะของ *N* ก็ต่อเมื่อ *N/M* เป็นเอ็นออลโมฟิลด์ และ *P* เป็นไอดีลเฉพาะของ *N* ก็ต่อเมื่อ *N/P* เป็นแอลเออินกรัลโดเมน

**คำสำคัญ** เอ็นแอลเอริง ไอดีลเฉพาะ ไอดีลใหญ่สุด

**Abstract**

This study aimed to expand some of the results of Shah, Rehman and Raees (2011). It defined and investigated the properties of prime ideals and maximal ideal of an nLA-ring. Moreover, *M* is a maximal ideal if and only if *N/M* is an n-almost field and *P* is a prime ideal of *N* if and only if *N/P* is an LA-integral domain.

**Keywords:** nLA-ring: Prime ideals: Maximal ideal

## 1. Introduction

Kazim and Naseeruddin [1] have introduced a pseudo associate law or left invertive law in a groupoid $G$ by

$(ab)c = (cb)a$ for all $a, b, c \in G$

and have named it as left invertive law, and called the groupoid a left almost semigroup (abbreviated as an LA-semigroup) if it satisfies left invertive law. The groupoid $G$ is also called an Abel-Grassmann's groupoids (abbreviated as an AG-groupoids), see [1] or [2]. It is a nonassociative algebraic structure midway between a groupoid and a commutative semigroup.

Kazim and Naseeruddin [1] asserted that, in every LA-semigroups $G$ a medial law hold

$(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in G$.

Mushtab and Khan [3] asserted that, in every LA-semigroups $G$ with left identity

$(ab)(cd) = (db)(ca)$ for all $a, b, c, d \in G$.

Further Khan, Faisal, and Amjid [4], asserted that, if a LA-semigroup $G$ with left identity the following law holds

$a(bc) = b(ac)$ for all $a, b, c \in G$.

Sarwar (Kamran) [4] defined LA-group as the following; a groupoid $G$ is called a left almost group, abbreviated as LA-group, if

(1) there exists $e \in G$ such that $ea = a$ for all $a \in G$,

(2) for every $a \in G$ there exists $a^{-1} \in G$ such that, $a^{-1}a = e$,

(3) $(ab)c = (cb)a$ for all $a, b, c \in G$.

Yusuf [5] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set R with two binary operations "+" and "•" is called a left almost ring, if $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup and distributive laws of "•" over "+" holds. Shah and Rehman [6] asserted that a commutative ring $(R, +, \cdot)$ we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining, for $a, b \in R$, $a \oplus b = b - a$ and $ab$ is same as in the ring. We cannot assume the addition to be commutative in an LA-ring.

An LA-ring $(R, +, \cdot)$ is said to be LA-integral domain if $ab = 0$ for all $a, b \in R$ then $a = 0$ or $b = 0$.

Let $(R, +, \cdot)$ be an LA-ring and $S$ be a non-empty subset of $R$ and $S$ is itself and LA-ring under the binary operation induced by $R$, the $S$ is called an LA-subring of $R$, then $S$ is called an LA-subring of $(R, +, \cdot)$. If $S$ is an LA-subring of an LA-ring $(R, +, \cdot)$ then $S$ is called a left ideal of R if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner.

By [7] a near-ring is a non-empty set N together with two binary operations "+" and "•" such that $(N, +)$ is a group (not necessarily abelian), $(N, +)$ is a semigroup and one sided distributive (left or right) of "•"over "+" holds.

By [8] An ideal $P$ of a near ring $N$ is said to be prime if $IJ \subseteq P$ then $I \subseteq P$ or $J \subseteq P$ for all $I, J$ ideal of $N$. Let $N$ be a near-ring, $K$ an ideal of $N$. Let $N/K = \{K+ n | n \in N\}$ be the set of cosets of $K$ in $N$. Then $(N/K, +, \cdot)$ is called the quotient near-ring of $N$ by or over $K$, where "+" and "•" are defined by $(K+ n_1)+(K + n_2) = K + n_1 + n_2$ and $(K+n_1)\cdot(K+n_2) = K+ n_1 n_2$ for all $n_1, n_2 \in N$.

## 2 Near Left Almost Rings

T. Shah, F. Rehman and M. Raees [8] introduces the concept of a near left almost ring (nLA-ring).

**Definition 2.1**. [9]. A non-empty set N with two binary operation "+" and "•" is called a near left almost ring (or simply an nLA-ring) if and only if

(1) $(N, +)$ is an LA-group.

(2) $(N, \cdot)$ is an LA-semigroup.

(3) Left distributive property of $\cdot$ over + holds, that is $a(b+c) = ab+ ac$ for all $a, b, c \in N$.

**Proposition 2.2** [9]. If $(N, +, \cdot)$ is an nLA-ring with additive left identity 0, then for all $a, b \in N$ we have.

(1) $0a = 0$.

(2) $a(-b) = -ab$.

**Definition 2.3.** [9]. An nLA-ring $(N, +, \cdot)$ with left identity 1; such that $1a = a$ for all $a \in N$, is called an nLA-ring with left identity.

**Definition 2.4**. [9]. A nonempty subset S of an nLA-ring $N$ is said to be an nLA-subring if and only if $S$ is itself an nLA-ring under the same binary operations as in $N$.

**Definition 2.5.** [9]. An nLA-subring $I$ of an nLA-ring $N$ is called a left ideal of $N$ if $NI \subseteq I$, and $I$ is called a right ideal if for all $n, m \in N$ and $i \in I$ such that $(i + n)m - nm \in I$; and is called two sided ideal or simply ideal if it is both left and right ideal.

**Definition 2.6** [9]. Let $N$ be an nLA-ring and $I$ be an ideal of $N$. Let $N/I = \{[n] = I+ n | n \in N\}$. Then $(N/I, +, \cdot)$ is called the quotient nLA-ring if $(I + n)+ (I + m) = I + n + m$ and $(I + n)\cdot(I + m) = I+ nm$ where $I+ n, I + m \in N/I$.

**Definition 2.7** [9]. Let $(N, +, \cdot)$ be an nLA-ring.

(a) An element $a \in N$ is called left (right) cancellative if $ab = ac$, then $b = c$ ($ba = ca$ then $b = c$) where $a, b, c \in N$, and $a$ is called cancellative if it is both left and right cancellative. However the nLA-ring $N$ is called cancellative if each element in $N$ is cancellative.

(b) A non-zero element $a$ of $N$ is called a left zero divisor if there exists $0 \neq b \in N$ such that $ab = 0$. Similarly $a$ is a right zero divisor if $ba = 0$. If $a$ is both a left and a right zero divisor, then $a$ is called a zero divisor.

**Definition 2.8** [9]. An nLA-ring $(D, +, \cdot)$ with left identity 1, is called an nLA-integral domain if it has no left zero divisor.

The theorem following we will proof right cancellation law hold in an nLA-integral domain by proof analogous as in [10].

**Theorem 2.9.** The right cancellation law holds in an nLA-ring $N$, then $N$ is an nLA-integral domain.

**Proof.** Let N is an nLA-ring with right cancellation and suppose that $ab = 0$ for some $a,b \in N$.

To show that $a = 0$. For $b \neq 0$. Consider

$ab = 0$

$ab = 0b$, by Proposition 2.2 (1)

$a = 0$, by right cancellation law.

Then $N$ is an nLA-ring integral domain.

**Definition 2.10** [9]. An nLA-ring $(F, +, \cdot)$ with left identity 1, is called a near almost field (n-almost field) if and only if each non-zero element of F has inverse under "×".

**Theorem 2.11** [9]. Every finite nLA-integral domain is an n-almost field.

**Definition 2.12** [9] Let $(N, +, \cdot)$ be an nLA-ring with left identity 1. An element $0 \neq n \in N$ is called unit if there exists $m \in N$ such that $mn = mn = 1$.

Next we study properties of unit of nLA-ring with proved same in [12].

**Lemma 2.13** If $N$ is an n-LAring with unity and $I$ is an ideal of $N$. Then $I = N$ if and only if $N$ contains a unit.

**Proof.** If I = N then $1_N \in$ N, and so N contains a unit.

Conversely, let $u \in$ I where u be a unit in N, then $u^{-1} \in$ N. Thus $u^{-1}u = 1_N \in$ N. Since I is an ideal we have for all $n \in N$, $n = n1_N \in N$ implies $N \subseteq I$. Thus I = N.

## 3. Maximal and Prime ideal of Near Left Almost Rings

Next we defines of a maximal and prime ideal in nLA-ring is defines the same as a maximal and prime ideal in near-ring in [7].

**Definition 3.1.** An ideal $M \neq N$ in nLA-ring $N$ is called a maximal ideal of $N$ if wherever $I$ is an ideal of $N$ such that $M \subseteq I \subseteq N$ then either $N = I$ or $M = I$.

**Definition 3.2** An ideal $P$ of a nLA-ring $N$ is said to be prime if $IJ \subseteq P$ then $I \subseteq P$ or $J \subseteq P$ for all $I$, $J$ ideal of $N$.

The following theorems with proved is analogous as in [12].

**Theorem 3.3.** Let $N$ be an nLA-ring with left identity and let $M$ be an ideal of $N$. Then $M$ is a maximal ideal if and only if $N/M$ is an n-almost field.

**Proof.** ($\Longrightarrow$) Suppose that $M$ is a maximal ideal in $N$ implies $M \neq N$. Let $(M + a) \in N/M$ with $a \notin M$ so that $M + a$ is not additive identity of $N/M$. We must show that $M + a$ has a multiplication inverse in $N/M$. Let

$I = \{m + na | n \in N, m \in M\}$.

Then $I$ is an ideal of $N$, since

$(m_1 + n_1a)+ (m_2 + n_2a) = (m_1 + m_2)+ (n_1 + n_2)a \in I$,

while also,

$0 = 0 + 0a$ and $-(m + na) = (-m)+ (-n)a$ are in $I$.

Now

$n_1(m + na) = n_1m + (n_1n)a$,

show that $n_1(m + na) \in I$ for $n_1 \in N$ also. Thus $I$ is an ideal.

For $m \in M$, $m = m + 0a \in I$, show that $M \subseteq I$ and $a = 0 + 1a$ show that $a \in I$. But from above $a \notin M$. Hence $I$ is an ideal of $N$ property containing $M$, $(M \subseteq I)$. Since $M$ is a maximal, we must have $I = N$. In particular, $1 \in I$. Then by definition of

$I$ there is $b \in N$ and $m \in M$ such that $1 = m + ba$. Therefore

$M + 1 = M + (m + ba) = (M + b)(M + a)$

so $M + b$ is a multiplication inverse of $M + a$.

($\Longleftarrow$) Suppose that $N/M$ is an n-almost field. Then $M + 1_N \neq M + 0$ implies that $1_N \notin N$ and so $M \neq N$. If $I$ is ideal such that

$M \subset I$, let $a \in I - M$. Then $M + a$ has a multiplication inverse in N/M say $(M + a)(M + b) = M + 1_N$. Consequently

$M + ab = M + 1_N$ and $ab - 1_N = m \in M$, implies $1_N = ab - m \in I$ because $a \in I$, $m \in M \subset I$. That is $1_N \in I$ so I = N,

by Lemma 2.13.

**Theorem 3.4.** Let *N* be an nLA-ring with left identity. Then *P* is a prime ideal of *N* if and only if *N/P* is an LA-ring integral domain.

**Proof.** Assume that *P* is a prime ideal of *N* and $P + a$, $P + b \in N/P$. Then $(P + a)(P + b) = P$ for all $a,b \in N$. Thus $ab \in P$.

Since *P* is a prime we have $a \in P$ *or* $b \in P$. Then $P + a = P$ *or* $P + b = P$. Hence *N/P* is an nLA-integral domain.

Conversely, assume that *N/P* is an nLA-integral domain with $ab \in P$. for all $a,b \in N$. Then $P + ab = P$ so

$(P + a)(P + b) = P$. Since *N/P* is an nLA-integral domain we have $P + a = P$ or $P + b = P$. Then $a \in P$ or $b \in P$. Thus *P* is a prime ideal of *N*.

**Corollary 3.5.** Let *N* be an nLA-ring with left identity and if *N* is a finite, then every prime ideal is maximal ideal.

**Proof** Let *P* be a prime ideal of *N* and if *N* be a finite. Then by Theorem 3.4 *N/P* is an nLA-ring integral domain and by Theorem 3.3, *N/P* is an n-almost field. Thus *P* is a maximal ideal of *N*.

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