The Journal of Applied Science **Vol. 19 No. 1: 36-50 [2020]**
วารสารวิทยาศาสตร์ประยุกต์ doi: 10.14416/j.appsci.2020.01.004
ISSN 1513-7805 Printed in Thailand ISSN 2586-9663 (Online)

Research Article

Asymptotic confidence ellipses for the re-parameterized inverse Gaussian distribution

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Received: 27/04/2020; Revised: 11/06/2020; Accepted: 18/06/2020

Abstract

The re-parameterized inverse Gaussian distribution is a very useful distribution for statistics and it is applied to various fields, such as physics, engineering, biology, etc. It is also appropriate to analyze the right-skewed data. In this research, we were interested in studying the Fisher-information matrix and we wanted to find the covariance matrix to form asymptotic confidence ellipses of parameters for the re-parameterized inverse Gaussian distribution. We compared the coverage probability with the confidence coefficient of 0.98 of confidence ellipses with sample sizes of n = 10, 30, 50, 100, 1,000, and 10,000. The parameters of λ = 0.5, 1, 5, 10, 50 and θ = 0.5, 1, 5, 10, 50. The data were simulated by the composition method which they were repeated 10,000 rounds in each case with RStudio programming. The simulation results indicate that the value of coverage probability had been close to the confidence coefficient of 0.98 at the sample size of 10,000.

Keywords: inverse Gaussian distribution, confidence ellipses, Fisher-information matrix, covariance matrix

Introduction

The inverse Gaussian distribution became known in statistics in the twentieth century from the research of Schrödinger (1915) and Tweedie (1957). This distribution is an alternative distribution suitable for an analysis of the right-skewed data. Therefore, it is popular and is applied to various fields such as physics, engineering, biology, finance, etc. As Folks & Chhikara (1978) explained in their research, the inverse Gaussian distribution is the continuous probability distribution which the probability density functions of inverse Gaussian distribution contain two parameters are μ and λ.

$$f_{IG}(x,\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\} \quad ;x>0, \tag{1}$$

where $\mu > 0$ and $\lambda > 0$, they represent a symbol of $X \sim IG(\mu,\lambda)$. And μ is a location parameter or an average parameter, λ is a shape parameter. Thus, the average of distribution is $E(X)=\mu$ and the variance of distribution is $Var(X)=\mu^3/\lambda$. Furthermore, the skewness is

$3\sqrt{\mu/\lambda}$. The maximum likelihood estimator of μ is $\tilde{\mu}=\bar{x}$ and the maximum likelihood estimator of λ is $\tilde{\lambda}=\left(\sum_{i=1}^{n}x_i^{-1}/n-1/\bar{x}\right)^{-1}$, where $\bar{x}=\sum_{i=1}^{n}x_i/n$.

Besides, the inverse Gaussian distribution still has a similar property with and bears a relation to the Birnbaum-Saunders distribution which was first introduced by Birnbaum & Saunders (1969). This Birnbaum-Saunders distribution was the failure distribution and was often used for reliability analysis. Afterward, the parameters of the Birnbaum-Saunders distribution were again adapted by Ahmed et al. (2008) in order to be suitable for use. Furthermore, they solved the relationship equation between the latest parameters and former parameters, including various properties of this distribution. Jangphanish & Budsaba (2013) considered the parameter estimation of the inverse Gaussian distribution based upon new parameters as Ahmed et al. (2008).

Therefore, the probability density function of the inverse Gaussian distribution with the new parameters can be written in this formula:

$$f_{IG}\left(x;\lambda,\theta\right)=\frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{3}{2}}exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^{2}\right]\quad ;x>0, \tag{2}$$

where $\lambda>0$ and $\theta>0$, they represent a symbol of $X\sim IG(\lambda,\theta)$, λ is the thickness of the sample, and θ is the nominal treatment loading on the sample. The maximum likelihood estimators of λ is $\hat{\lambda}=\dfrac{n}{T\bar{x}-n}$ and the maximum likelihood estimators of θ is $\hat{\theta}=\dfrac{T\bar{x}^2-\sum_{i=1}^{n}x_i}{n}$ where $T=\sum_{i=1}^{n}\dfrac{1}{x_i}$.

The two types of estimation of population parameters are point estimation and interval estimation. The estimation of the population parameters, appealing to a study due to containing a single statistical value, is the point estimation. The interval estimation or the confidence interval estimation is the estimation of interesting population parameters because it contains an interval value which is between two values by containing the point estimation at the center between the two values.

In the past decade, numerous studies have given special attention to the confidence interval for one-dimensional space; few researchers are interested in studying the confidence interval for two-dimensional space (which is called the elliptical confidence interval).

From the previous research into the inverse Gaussian distribution, we found that there was no research work on the confidence interval of asymptotic ellipses for the re-parameterized inverse Gaussian distribution. Consequently, we are interested in doing a study into the confidence interval of asymptotic ellipses for the re-parameterized inverse Gaussian distribution based on distribution as Jangphanish & Budsaba (2013).

A brief review of the re-parameterized inverse Gaussian distribution

The basic parameters of the re-parameterized inverse Gaussian distribution are λ and θ. This distribution is a group of continuous probability distributions with values of random variables from 0 to ∞. The probability density functions of the random variable with the re-parameterized inverse Gaussian distribution that proposed by Jangphanish & Budsaba (2013) is

$$f\left(x;\lambda,\theta\right) = \frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{3}{2}}\exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}}-\sqrt{\frac{x}{\theta}}\right)^{2}\right\}\ ;\ x>0, \tag{3}$$

where $\lambda > 0$ is the thickness of the machine element and $\theta > 0$ is the nominal treatment pressure on the machine.

The re-parameterized inverse Gaussian distributions, which consists of two parameters, are λ and θ. The likelihood functions of the re-parameterized inverse Gaussian distributions depend on the two parameters. They can be written in this formula:

$$L\left(x;\lambda,\theta\right) = \frac{\lambda^{n}\theta^{\frac{n}{2}}}{\left(\sqrt{2\pi}\right)^{n}\prod_{i=1}^{n}x_i^{\frac{3}{2}}}\exp\left\{-\sum_{i=1}^{n}\frac{\lambda^{2}\theta}{2x_i}+n\lambda-\frac{\sum_{i=1}^{n}x}{2\theta}\right\}. \tag{4}$$

Therefore, the logarithm of the probability function can be written in this formula:

$$l\left(x;\lambda,\theta\right) = \ln L\left(x;\lambda,\theta\right) = n\ln\lambda+\frac{n}{2}\ln\theta-n\ln\sqrt{2\pi}-\sum_{i=1}^{n}x_i^{\frac{3}{2}}-\frac{\lambda^{2}\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}+n\lambda-\frac{\sum_{i=1}^{n}x}{2\theta},$$

and the maximum likelihood estimators of parameters λ and θ can be written in this form

$$\hat{\theta} = \frac{T\bar{X}^{2}-\sum_{i=1}^{n}x_i}{n} \quad \text{and} \quad \hat{\lambda} = \frac{n}{T\bar{X}-n}, \tag{5}$$

where $T = \sum_{i=1}^{n}\frac{1}{x_i}$, $\bar{X} = \frac{\sum_{i=1}^{n}x_i}{n}$,

and $E(X)$ is the first moment of the re-parameterized inverse Gaussian distribution, therefore it is in this form: $E(X) = \lambda\theta$,

and
$$\begin{aligned} Var\left(X\right) &= E\left(X^{2}\right)+\left(E\left(X\right)\right)^{2} \\ &= \lambda\theta^{2}\left(\lambda+1\right)-\lambda^{2}\theta^{2} \\ &= \lambda^{2}\theta^{2}+\lambda\theta^{2}-\lambda^{2}\theta^{2} = \lambda\theta^{2}. \end{aligned} \tag{6}$$

Theoretical results

1. The Fisher information matrix for the re-parameterized inverse Gaussian distribution

Give a random variable X to be the re-parameterized inverse Gaussian distribution with parameters $\left(\lambda,\theta\right)$. The Fisher Information Matrix is Θ when $\Theta = \left(\mu,\lambda\right)$ is a two-dimensional vector of the parameter standing for $I\left(\Theta\right)$ as shown in below

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$$I(\Theta) = I(\lambda,\theta) = -E\begin{bmatrix} \frac{\partial^2}{\partial\lambda^2}\ln f(x;\lambda,\theta) & \frac{\partial^2}{\partial\lambda\partial\theta}\ln f(x;\lambda,\theta) \\ \frac{\partial^2}{\partial\theta\partial\lambda}\ln f(x;\lambda,\theta) & \frac{\partial^2}{\partial\theta^2}\ln f(x;\lambda,\theta) \end{bmatrix}. \tag{7}$$

The logarithm of the probability function of the re-parameterized inverse Gaussian distribution is

$$\ln f(x;\lambda,\theta) = \ln\left[\frac{\lambda\theta^{\frac{1}{2}}}{\sqrt{2\pi}x^{\frac{3}{2}}}\exp\left\{-\frac{\lambda^2\theta}{2x}+\lambda-\frac{x}{2\theta}\right\}\right]$$

$$= \ln(\lambda)+\frac{1}{2}\ln(\theta)-\ln\sqrt{2\pi}-\ln\left(x^{\frac{3}{2}}\right)-\frac{\lambda^2\theta}{2x}+\lambda-\frac{x}{2\theta}. \tag{8}$$

From $E(X)=\lambda\theta$, thus we can estimate the expectation of reciprocal $E(1/X)$ by Taylor expansion around $E(X)$ of a variable, so we obtain

$$E\left(\frac{1}{X}\right) \approx E\left(\frac{1}{E(X)}-\frac{1}{E(X)^2}(X-E(X))+\frac{1}{E(X)^3}(X-E(X))^2\right)$$

$$= \frac{1}{E(X)}+\frac{1}{E(X)^3}Var(X)$$

$$= \frac{1}{\lambda\theta}+\frac{1}{\lambda^3\theta^3}\lambda\theta^2 = \frac{1}{\lambda\theta}+\frac{1}{\lambda^2\theta} = \frac{1}{\lambda^2\theta}(\lambda+1). \tag{9}$$

Hence, $E\begin{bmatrix} \frac{\partial^2}{\partial\lambda^2}\ln f(x;\lambda,\theta) & \frac{\partial^2}{\partial\lambda\partial\theta}\ln f(x;\lambda,\theta) \\ \frac{\partial^2}{\partial\theta\partial\lambda}\ln f(x;\lambda,\theta) & \frac{\partial^2}{\partial\theta^2}\ln f(x;\lambda,\theta) \end{bmatrix}$ will be in these formulas:

$$E\left(\frac{\partial^2}{\partial\theta^2}\ln f(x;\lambda,\theta)\right) = -\frac{1}{2\theta^2}-\frac{E(x)}{\theta^3} = -\frac{1}{2\theta^2}-\frac{\lambda\theta}{\theta^3}$$

$$= -\frac{1}{2\theta^2}-\frac{\lambda}{\theta^2} = -\frac{(2\lambda+1)}{2\theta^2} \tag{10}$$

$$E\left(\frac{\partial^2}{\partial\lambda^2}\ln f(x;\lambda,\theta)\right) = -\frac{1}{\lambda^2}-\theta E\left(\frac{1}{x}\right)$$

$$= -\frac{1}{\lambda^2}-\theta\left(\frac{1}{\lambda^2\theta}\right)(\lambda+1)$$

$$= -\frac{1}{\lambda^2}-\frac{\theta}{\lambda\theta}-\frac{\theta}{\lambda^2\theta} = -\frac{1}{\lambda^2}-\frac{1}{\lambda}-\frac{1}{\lambda^2}$$

$$= -\frac{1}{\lambda}-\frac{2}{\lambda^2} = -\frac{1}{\lambda^2}(\lambda+2) \tag{11}$$

doi: 10.14416/j.appsci.2020.01.004

$$E\left(\frac{\partial^2}{\partial\lambda\partial\theta}\ln f(x;\lambda,\theta)\right)=-\lambda E\left(\frac{1}{x}\right)$$

$$=-\frac{1}{\lambda\theta}(\lambda+1) \tag{12}$$

Therefore, the Fisher Information Matrix for the re-parameterized inverse Gaussian distribution is in this form as shown.

$$I(\Theta)=-\begin{bmatrix}-\frac{1}{\lambda^2}(\lambda+2) & -\frac{(\lambda+1)}{\lambda\theta}\\ -\frac{(\lambda+1)}{\lambda\theta} & -\frac{(2\lambda+1)}{2\theta^2}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{(\lambda+2)}{\lambda^2} & \frac{(\lambda+1)}{\lambda\theta}\\ \frac{(\lambda+1)}{\lambda\theta} & \frac{(2\lambda+1)}{2\theta^2}\end{bmatrix}. \tag{13}$$

When Θ is the two-dimensional vector of parameters and $\Theta=(\lambda,\theta)$, so we will obtain the Fisher information matrix for sample sizes n are

$$I_n(\Theta)=\begin{bmatrix}\frac{n(\lambda+2)}{\lambda^2} & \frac{n(\lambda+1)}{\lambda\theta}\\ \frac{n(\lambda+1)}{\lambda\theta} & \frac{n(2\lambda+1)}{2\theta^2}\end{bmatrix}. \tag{14}$$

2. The covariance matrix

The value of the covariance matrix and the value of the inverse of the Fisher information matrix are equal. They stand for $\Lambda=I_n^{-1}(\Theta)=I_n^{-1}(\lambda,\theta)$. Consequently, when n comes to ∞, it will be the following equations

$$I_n^{-1}(\Theta)=\frac{1}{\left(\frac{n(\lambda+2)}{\lambda^2}\times\frac{n(2\lambda+1)}{2\theta^2}\right)-\left(\frac{n(\lambda+1)}{\lambda\theta}\times\frac{n(\lambda+1)}{\lambda\theta}\right)}\begin{bmatrix}\frac{n(2\lambda+1)}{2\theta^2} & -\frac{n(\lambda+1)}{\lambda\theta}\\ -\frac{n(\lambda+1)}{\lambda\theta} & \frac{n(\lambda+2)}{\lambda^2}\end{bmatrix}$$

$$=\frac{1}{\frac{n^2(2\lambda+1)(\lambda+2)}{2\lambda^2\theta^2}-\frac{n^2(\lambda+1)^2}{\lambda^2\theta^2}}\begin{bmatrix}\frac{n(2\lambda+1)}{2\theta^2} & -\frac{n(\lambda+1)}{\lambda\theta}\\ -\frac{n(\lambda+1)}{\lambda\theta} & \frac{n(\lambda+2)}{\lambda^2}\end{bmatrix}. \tag{15}$$

Consider this formula

$$\frac{n^2(2\lambda+1)(\lambda+2)}{2\lambda^2\theta^2}-\frac{2}{2}\times\frac{n^2(\lambda+1)^2}{\lambda^2\theta^2}=\frac{n^2(2\lambda+1)(\lambda+2)}{2\lambda^2\theta^2}-\frac{2n^2(\lambda+1)^2}{2\lambda^2\theta^2}$$

$$= \frac{n^2(2\lambda+1)(\lambda+2) - 2n^2(\lambda+1)^2}{2\lambda^2\theta^2}$$

$$= \frac{n^2\left[2\lambda^2+4\lambda+\lambda+2\right]}{2\lambda^2\theta^2} - \frac{2n^2\left[\lambda^2+2\lambda+1\right]}{2\lambda^2\theta^2}$$

$$= \frac{n^2\left[2\lambda^2+5\lambda+2\right] - 2n^2\left[\lambda^2+2\lambda+1\right]}{2\lambda^2\theta^2}$$

$$= \frac{2n^2\lambda^2+5n^2\lambda+2n^2-2n^2\lambda^2-4n^2\lambda-2n^2}{2\lambda^2\theta^2}$$

$$= \frac{2n^2\lambda^2-2n^2\lambda^2+5n^2\lambda-4n^2\lambda+2n^2-2n^2}{2\lambda^2\theta^2}$$

$$= \frac{n^2\lambda}{2\lambda^2\theta^2} = \frac{n^2}{2\lambda\theta^2}. \quad (16)$$

That is

$$\Lambda = I_n^{-1}(\Theta) = \frac{2\lambda\theta^2}{n^2}\begin{bmatrix} \frac{n(2\lambda+1)}{2\theta^2} & -\frac{n(\lambda+1)}{\lambda\theta} \\ -\frac{n(\lambda+1)}{\lambda\theta} & \frac{n(\lambda+2)}{\lambda^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\lambda(2\lambda+1)}{n} & -\frac{2\theta(\lambda+1)}{n} \\ -\frac{2\theta(\lambda+1)}{n} & \frac{2\theta^2(\lambda+2)}{n\lambda} \end{bmatrix}. \quad (17)$$

3. An asymptotic normal distribution

Consider the order of random variable from maximum likelihood $\hat{\Theta}_n^{(MLE)}$ using the Delta method. It will be this form

$$\sqrt{n}\left(\hat{\Theta}_n^{(MLE)} - \Theta\right) \xrightarrow{d} X \sim N_2\left(0, I^{-1}(\Theta)\right)$$

$$\frac{\left(\hat{\Theta}_n^{(MLE)} - \Theta\right)}{\sqrt{I_n^{-1}(\Theta)}} \xrightarrow{d} Z \sim N_2\left(0, I_2\right).$$

From the value of the inverse of the Fisher information matrix is equal to the value of the covariance matrix, it will be

$$\frac{\left(\hat{\Theta}_n^{(MLE)} - \Theta\right)}{\sqrt{\Lambda}} \xrightarrow{d} N_2\left(0, I_2\right). \quad (18)$$

Therefore,

 doi: 10.14416/j.appsci.2020.01.004

$$\left(\Lambda^{\frac{1}{2}}\right)^{-1}\left(\hat{\Theta}_n^{(MLE)}-\Theta\right)\xrightarrow{d} Z \sim N_2\left(0,I_2\right), \tag{19}$$

where $I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is an identity matrix and Z_1, Z_2 are random variables with an average normal distribution of 0 and the variance of 1, and they are independent. Thus, $Z'=(Z_1,Z_2)$ is the bivariate normal distribution $N_2(0,I_2)$ as it shows

$$\left(\hat{\Theta}_n^{(MLE)}-\Theta\right)'\left(\Lambda\right)^{-\frac{1}{2}}\left(\Lambda\right)^{-\frac{1}{2}}\left(\hat{\Theta}_n^{(MLE)}-\Theta\right)=\left(\hat{\Theta}_n^{(MLE)}-\Theta\right)'\Lambda^{-1}\left(\hat{\Theta}_n^{(MLE)}-\Theta\right)$$
$$=Z'Z=Z_1^2+Z_2^2=\sum_{i=1}^{2}Z_i^2 .$$

This distribution is Chi-Square distribution with two degrees of freedom (χ_2^2).

4. The region of the confidence for the re-parameterized inverse Gaussian distribution

From Duangchana & Budsaba (2014), Hansoongnern et al. (2018) and Phaphan & Pongsart (2019) we will obtain the confidence region of $100(1-\alpha)\%$ for the parameter Θ of Two-Dimensional Normal Distribution. It is an ellipse as shown in this equation.

$$\left(\Theta_n^{(MLE)}-\Theta\right)'\Lambda^{-1}\left(\hat{\Theta}_n^{(MLE)}-\Theta\right)\le\chi_2^2\left(\alpha\right). \tag{20}$$

Since covariance matrix of the re-parameterized inverse Gaussian distribution is

$$\Lambda^{-1}=I_n(\Theta). \tag{21}$$

We will obtain the confidence region of $100(1-\alpha)\%$ for the parameter $\Theta'=(\lambda,\theta)$ of the re-parameterized inverse Gaussian distribution as it indicates

$$\begin{bmatrix}\hat{\lambda}_n^{(MLE)}-\lambda & \hat{\theta}_n^{(MLE)}-\theta\end{bmatrix}\begin{bmatrix}\frac{n(\lambda+2)}{\lambda^2} & \frac{n(\lambda+1)}{\lambda\theta}\\ \frac{n(\lambda+1)}{\lambda\theta} & \frac{n(2\lambda+1)}{2\theta^2}\end{bmatrix}\begin{bmatrix}\hat{\lambda}_n^{(MLE)}-\lambda\\ \hat{\theta}_n^{(MLE)}-\theta\end{bmatrix}\le\chi_{(2)}^2\left(\alpha\right) \tag{22}$$

$$\frac{n(\lambda+2)}{\lambda^2}\left(\hat{\lambda}_n^{(MLE)}-\lambda\right)^2+\frac{2n(\lambda+1)}{\lambda\theta}\left(\hat{\lambda}_n^{(MLE)}-\lambda\right)\left(\hat{\theta}_n^{(MLE)}-\theta\right)+\frac{n(2\lambda+1)}{2\theta^2}\left(\hat{\theta}_n^{(MLE)}-\theta\right)^2\le\chi_{(2)}^2\left(\alpha\right). \tag{23}$$

Simulation results

In this topic, we studied the coverage probability of asymptotic confidence ellipses for the re-parameterized inverse Gaussian distribution. We defined parameter of $\lambda=$ 0.5, 1, 5, 10, 50 and parameters of $\theta=$ 0.5, 1, 5, 10, 50 by simulating random numbers of the re-parameterized inverse Gaussian distribution with the composition method (Ngamkham, 2019) of the sample sizes of $n=$ 10, 30, 50, 100, 1,000, 10,000. We took the random numbers which we obtained to

 doi: 10.14416/j.appsci.2020.01.004

estimate the parameters of λ and θ with the maximum likelihood method (Jangphanish & Budsaba, 2013), and afterward they were redone 10,000 rounds in each case. Lastly, asymptotic confidence ellipses were constructed and calculated the value of coverage probability of asymptotic confidence ellipses with the confidence coefficient of 0.98 by using RStudio programming as indicated in Tables 1-6 and Figures 1-6)Here it shows some cases(.

Table 1. A sample size of $n = 10$, an average of parameter estimates of λ, an average of parameter estimates of θ and the value of Coverage Probability (CP)

λ	θ	The average estimates		CP
		$\hat{\lambda}$	$\hat{\theta}$	
0.5	0.5	0.8675	0.5452	0.5823
	1	0.8575	1.0897	0.5937
	5	0.8545	5.4169	0.5906
	10	0.8719	10.3855	**0.5974**
	50	0.8508	54.4864	0.5907
1	0.5	1.5615	0.5036	0.6373
	1	1.5807	0.9974	0.6337
	5	1.5689	4.938	0.6478
	10	1.5628	9.9506	0.6407
	50	1.5689	49.7005	**0.6502**
5	0.5	7.2783	0.4597	0.6436
	1	7.3671	0.9119	0.6457
	5	7.2711	4.5582	**0.6517**
	10	7.2696	9.1681	0.6509
	50	7.2986	45.7389	0.6506
10	0.5	14.4873	0.4558	0.6105
	1	14.5794	0.9016	0.6151
	5	14.4262	4.5457	0.6084
	10	14.378	9.0936	0.618
	50	14.423	45.6088	**0.6182**
50	0.5	71.0387	0.4527	0.4894
	1	72.0167	0.9048	0.4791
	5	70.3464	4.5332	**0.4969**
	10	71.1927	9.0702	0.4932
	50	71.7472	44.921	0.4858

doi: 10.14416/j.appsci.2020.01.004

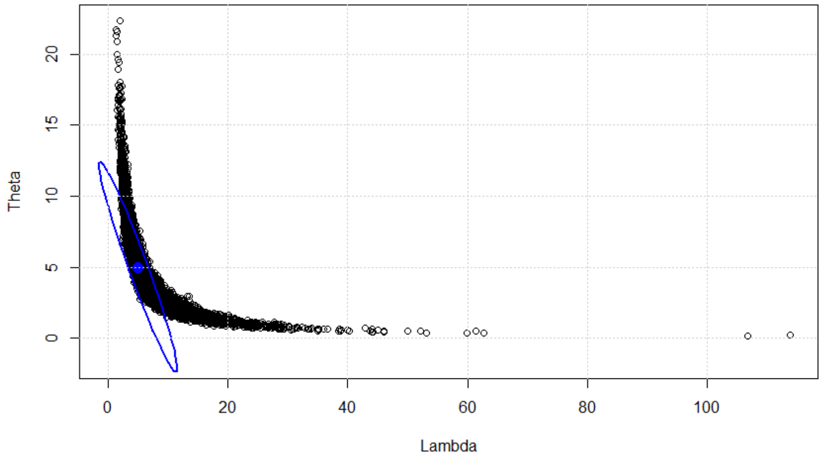

Figure 1. Asymptotic confidence ellipses in the case of $n=10,\ \lambda=5,\ \theta=5$

Table 2 . A sample size of $n=30$, an average of parameter estimates of λ, an average of parameter estimates of θ and the value of Coverage Probability (CP)

λ	θ	The average estimates		CP
		$\hat{\lambda}$	$\hat{\theta}$	
0.5	0.5	0.5905	0.5186	**0.7504**
	1	0.5937	1.0262	0.7474
	5	0.5914	5.1701	0.7484
	10	0.5956	10.2686	0.743
	50	0.5909	52.1257	0.7439
1	0.5	1.1473	0.4996	0.7897
	1	1.1474	0.9978	**0.7949**
	5	1.1508	4.9867	0.7892
	10	1.1518	9.9833	0.7917
	50	1.1459	50.0676	0.7916
5	0.5	0.4874	5.5757	0.8024
	1	5.5935	0.9745	0.7957
	5	5.568	4.8754	0.8055
	10	5.611	9.7014	0.7985
	50	5.5712	48.8262	**0.8056**
10	0.5	11.2136	0.4822	0.7612
	1	11.1245	0.9716	0.7676
	5	11.1737	4.8381	0.7652
	10	11.105	9.7371	0.7672
	50	11.1026	48.6128	**0.7687**
50	0.5	55.5276	0.4852	0.6323
	1	55.5863	0.9677	**0.6355**
	5	55.6313	4.8275	0.635
	10	55.4025	9.7275	0.6293
	50	56.0297	47.9986	0.6304

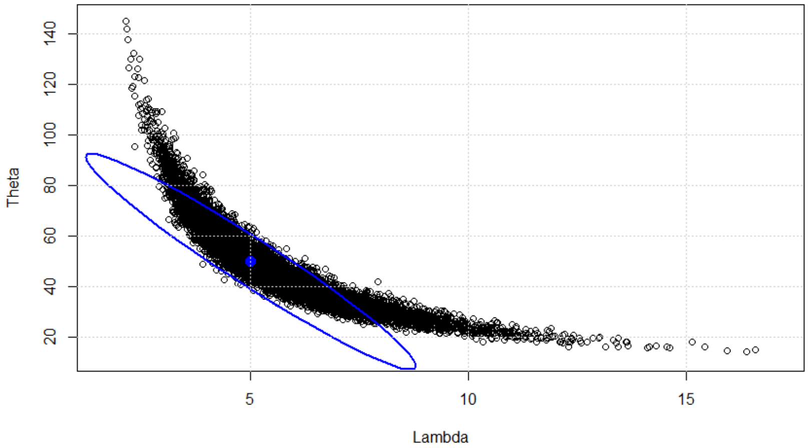

Figure 2. Asymptotic confidence ellipses in the case of $n = 30,\ \lambda = 5,\ \theta = 50$

Table 3 . A sample size of $n = 50$, an average of parameter estimates of λ, an average of parameter estimates of θ and the value of Coverage Probability (CP)

λ	θ	The average estimates		CP
		$\hat{\lambda}$	$\hat{\theta}$	
0.5	0.5	0.5529	0.513	0.7964
	1	0.5544	1.0175	**0.8063**
	5	0.5544	1.0175	**0.8063**
	10	0.5539	10.2085	0.7962
	50	0.5514	51.2268	0.8001
1	0.5	1.09	0.4969	0.84
	1	1.0868	1.001	0.8395
	5	1.0899	4.9634	**0.8409**
	10	1.0901	9.9562	0.8381
	50	1.0918	49.7151	0.8394
5	0.5	5.3281	0.4928	0.8463
	1	5.3495	0.9805	0.8478
	5	5.3338	4.9295	0.8468
	10	5.3388	9.8379	0.8522
	50	5.3318	49.2313	**0.8523**
10	0.5	10.6684	0.4902	0.8183
	1	10.6326	0.9857	0.8159
	5	10.6656	4.91	**0.823**
	10	10.6833	9.8181	0.8114
	50	10.6307	49.2032	0.8219
50	0.5	53.2004	0.4907	0.691
	1	53.2717	0.9796	0.6989
	5	53.1505	4.9103	0.6975
	10	53.3072	9.7789	**0.7016**
	50	53.3408	48.9673	0.6922

doi: 10.14416/j.appsci.2020.01.004

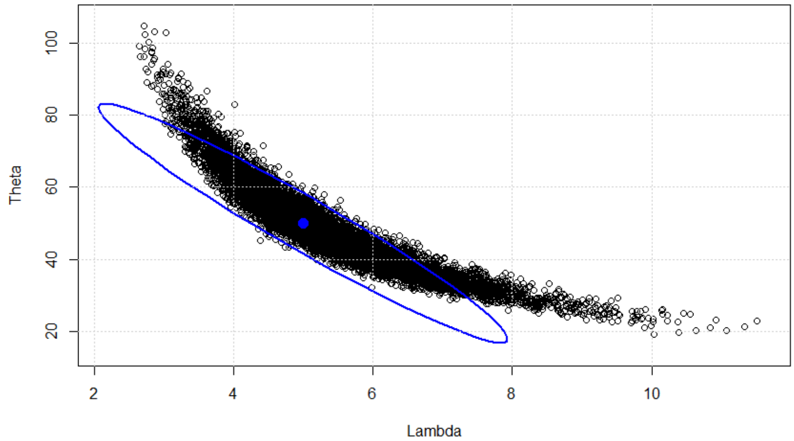

Figure 3. Asymptotic confidence ellipses in the case of $n=50,\ \lambda=5,\ \theta=50$

Table 4 . A sample size of $n=100$, an average of parameter estimates of λ, an average of parameter estimates of θ and the value of Coverage Probability (CP)

λ	θ	The average estimates $\hat{\lambda}$	The average estimates $\hat{\theta}$	CP
0.5	0.5	0.5254	0.5052	0.8599
	1	0.528	1.0048	0.8579
	5	0.5278	5.0348	0.8547
	10	0.5249	10.1248	0.8659
	50	0.5248	50.6336	**0.8694**
1	0.5	1.0388	0.5013	**0.9001**
	1	1.0394	1.001	0.8976
	5	1.0419	4.9992	0.8939
	10	1.0423	10.0013	0.8894
	50	1.0415	49.9095	0.8994
5	0.5	5.1529	0.4974	0.8987
	1	5.1646	0.9922	0.8991
	5	5.1495	4.97	**0.9015**
	10	5.1658	9.9094	0.8984
	50	5.1621	49.6401	0.8967
10	0.5	10.2967	0.4969	0.8713
	1	10.3127	0.992	0.8733
	5	10.3233	4.9551	0.874
	10	10.3107	9.9127	**0.8848**
	50	10.3062	49.5991	0.8824
50	0.5	51.5636	0.4952	0.7704
	1	51.4611	0.9921	0.7747
	5	51.5803	4.9467	0.7748
	10	51.4586	9.9216	0.7716
	50	51.6396	49.429	**0.7787**

doi: 10.14416/j.appsci.2020.01.004

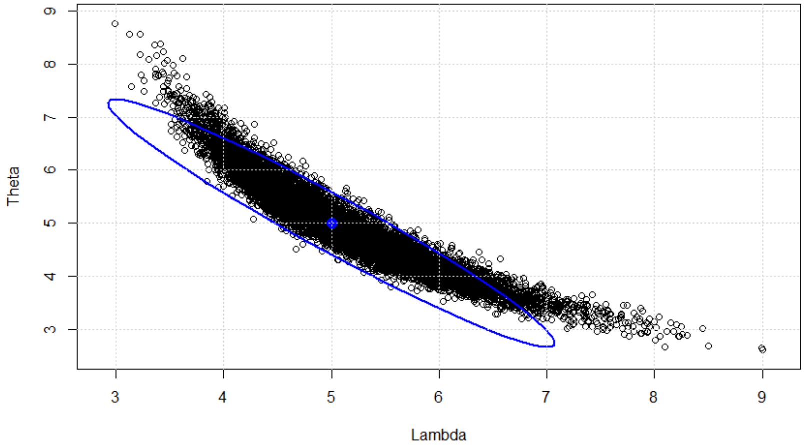

Figure 4. Asymptotic confidence ellipses in the case of $n = 100,\ \lambda = 5,\ \theta = 5$

Table 5 . A sample size of $n = 1,000$, an average of parameter estimates of λ, an average of parameter estimates of θ and the value of Coverage Probability (CP)

λ	θ	The average estimates		CP
		$\hat{\lambda}$	$\hat{\theta}$	
0.5	0.5	0.5023	0.5008	**0.9625**
	1	0.5026	1.0007	0.961
	5	0.5023	5.0072	0.9602
	10	0.5026	10.0084	0.9594
	50	0.5028	50.021	0.9596
1	0.5	1.0033	0.5003	**0.9685**
	1	1.0036	1.0008	0.9684
	5	1.0041	5.0027	0.9654
	10	1.0047	9.9927	0.9676
	50	1.0035	50.0385	0.9668
5	0.5	5.0171	0.4994	0.9668
	1	5.0161	0.9991	0.966
	5	5.0173	4.9944	0.9685
	10	5.0204	9.9834	0.9655
	50	5.0188	49.9296	**0.9687**
10	0.5	10.0401	0.4992	0.9616
	1	10.0263	0.9996	**0.9654**
	5	10.0251	4.9985	0.9634
	10	10.0278	9.9965	0.9627
	50	10.0287	49.9667	0.9651
50	0.5	50.1816	0.4992	0.9322
	1	50.1523	0.999	0.9301
	5	50.1736	4.9928	**0.9376**
	10	50.1553	9.9891	0.9328
	50	50.1492	49.9536	0.9316

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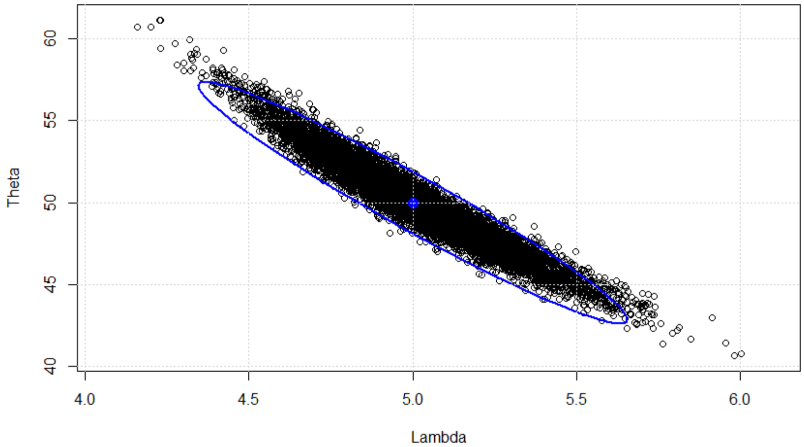

Figure 5. Asymptotic confidence ellipses in the case of $n=1{,}000,\ \lambda=5,\ \theta=50$

Table 6 . A sample size of $n=10,000$, an average of parameter estimates of λ, an average of parameter estimates of θ and the value of Coverage Probability (CP)

λ	θ	The average estimates		CP
		$\hat{\lambda}$	$\hat{\theta}$	
0.5	0.5	0.5003	0.4999	0.9776
	1	0.5001	1.0005	0.9759
	5	0.5002	5.0002	0.9776
	10	0.5003	9.9975	0.9768
	50	0.5004	49.985	**0.9807**
1	0.5	1.0006	0.4999	0.9772
	1	1.0005	0.9997	0.9783
	5	1.0002	5.0013	0.9787
	10	1.0005	9.9972	0.9787
	50	1.0005	49.9921	**0.9791**
5	0.5	5.0003	0.5001	0.9788
	1	5.0024	0.9998	**0.9799**
	5	5.0033	4.9981	0.9782
	10	5.0021	9.9982	0.977
	50	5.002	49.9908	0.9761
10	0.5	10.0004	0.5001	0.9762
	1	10.0025	1.0001	0.977
	5	10.002	5.0002	0.9787
	10	10.003	9.9994	0.9765
	50	10.0031	49.9937	**0.9788**
50	0.5	50.0165	0.4999	0.9758
	1	50.0279	0.9996	0.9734
	5	50.0095	5.0002	0.972
	10	50.0122	9.9997	0.9712
	50	50.0095	50.0144	**0.9773**

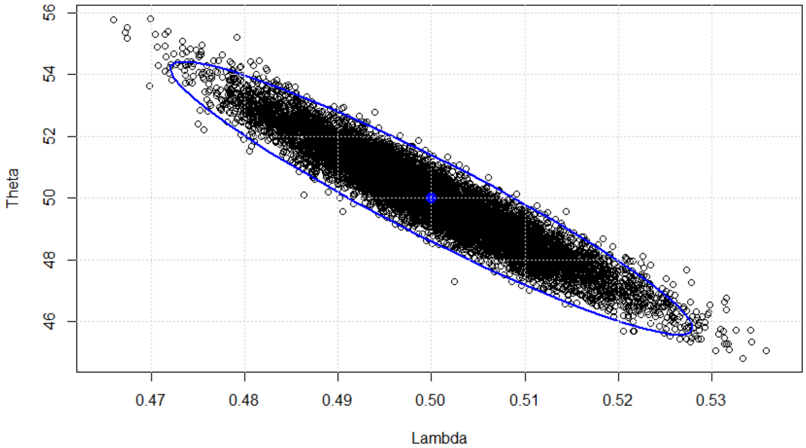

Figure 6. Asymptotic confidence ellipses in the case of $n = 10,000,\ \lambda = 0.5,\ \theta = 50$

Discussion

The research results of simulation as shows in Table 1-6 revealed that the parameter estimation of λ that had been introduced had more values than the actual values)overestimate(. Also, from Figure 2: $n = 30,\ \lambda = 5,\ \theta = 50$ showed that several parameter estimations came out of the ellipse, but when the sample size had increased to 1,000 as shown in Figure 5: $n = 1,000,\ \lambda = 5,\ \theta = 50$, the number of parameter estimations coming out of the ellipse decreased. Similarly, from Figure 1: $n = 10,\ \lambda = 5,\ \theta = 5$ were found that several parameter estimations came out of the ellipse, but when the sample size had increased to 10,000 as it indicates in Figure 6: $n = 10,000,\ \lambda = 5,\ \theta = 5$, the number of parameter estimations coming out of the ellipse reduced significantly.

The research indicated that the coverage probability of asymptotic confidence ellipses would have been increasing value when the sample size of n rose up. As we calculated, the value of coverage probability had been close to the confidence coefficient of 0.98 at the sample size of $n = 10,000$ which gave us the maximum coverage probability. Consequently, the value of coverage probability of asymptotic confidence ellipses for the re-parameterized inverse Gaussian distribution would have been increasing when the sample size was large.

In this research, we use the maximum likelihood estimates that have more errors when sample sizes are small. Therefore, the value of coverage probabilities are very low in the cases of small sample sizes.

Besides, for n less than or equal to 100, the parameter of $\lambda = 5$ and the various parameter of θ gave us the maximum coverage probability (as shown in Table 1-4). Therefore, the various parameter of λ were able to cause the value of coverage probability of asymptotic confidence ellipses for the re-parameterized inverse Gaussian distribution.

Conclusion

In the theoretical part, we calculate the Fisher information matrix for sample sizes n in closed form for a probability density function under the re-parameterized inverse Gaussian distribution and find the covariance matrix which equals an inverse of the Fisher information

 doi: 10.14416/j.appsci.2020.01.004

matrix for the sample size n to construct a normal approximation that gives elliptical confidence regions with center $\Theta = (\lambda, \theta)$ for all 150 cases.

For the simulation part, the objective of this part is conducted for an asymptotic confidence ellipses construction for parameters of the re-parameterized inverse Gaussian distribution and we compare the coverage probabilities of confidence ellipses of parameters for a re-parameterized inverse Gaussian distribution with the confidence coefficient 0.98. We consider the parameters of λ = 0.5, 1, 5, 10, 50, θ = 0.5, 1, 5, 10, 50 and the sample sizes n as 10, 30, 50, 100, 1,000, 10,000. The data were generated by a simulation technique using RStudio programming and the experimental is repeated 10,000 times to obtain the coverage probability. The simulation results indicated that the coverage probability of asymptotic confidence ellipses would have been increasing value when the sample size of n increase and the various values of λ can cause the coverage probability values of asymptotic confidence ellipses to increase or decrease. In connection with this research, we could continue applying to contemporary problems for real life data, mostly in Engineering. Also, we would like to suggest other distribution such as Weilbull distribution, log-normal distribution, extreme value distribution, Gompertz distribution and log-logistic distribution for constructing the confidence ellipses.

Acknowledgments

This research was funded by the Faculty of Applied Science, King Mongkut's University of Technology North Bangkok in 2019. Contract no. 6245101

References

Ahmed, S.E., Budsaba, K., Lisawadi, S., & Volodin, A. (2008). Parametric estimation for the Birnbaum-Saunders lifetime distribution based on new parametrization. *Thailand Statistician*, *6*(2), 213-240.

Birnbaum, Z. W., & Saunders, S. C. (1969). A new family of lifetime distribution. *Applied Probability*, *6*, 319-327.

Duangchana, N. & Budsaba, K. (2014). Asymptotic confidence ellipses of parameters for the inverse Gaussian distribution. *Thammasat International Journal of Science and Technology*, *19*(2), 22-29.

Folks, J. L. & Chhikara, R. S. (1978). The inverse Gaussian distribution and its statistical application - a review. *Journal of the Royal Statistical Society.Serie B*, *40*, 263–289.

Hansoongnern, K., Sakonthawichot, B., Sittisareesmer, A. & Phaphan, W. (2018). Estimation of asymptotic confidence ellipses for Birnbaum-Saunders distribution. *Burapha Science Journal*, *23(2)*, 1029-1043. (in Thai)

Jangphanish, K. & Budsaba, K. (2013). Parameter estimation for re-parameterized inverse Gaussian distribution. *Thammasat International Journal of Science and Technology*, *18*(1), 43-53.

Ngamkham, T. (2019). On the crack random numbers generation procedure. *Lobachevskii Journal of Mathematics*, *40*(8), 1204–1217.

Phaphan, W. & Pongsart, T. (2019). Asymptotic confidence ellipses for length-biased inverse Gaussian distribution. *The Journal of KMUTNB*, *29*(2), 332-341. (in Thai)

Schrödinger, E. (1915). Zur theorie der fall-und steigversuche an teilchen mit brownscher bewegung. *Physikalische Zeitschrift*, *16*, 289-295.

Tweedie, M.C.K. (1957). Statistical properties of inverse Gaussian distributions I. *The Annals of Mathematical Statistics*, *28*, 362–377.