

## Research Article

# On the Beta Topp-Leone Exponential Distribution

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Received: 09/08/2020; Revised: 18/01/2021; Accepted: 14/02/2021

## Abstract

In this work, a new generalized of the exponential distribution, called the beta Topp-Leone exponential distribution, is introduced. Mathematical properties of the proposed distribution are also provided, such as, some expansions of the probability density function and the cumulative distribution function, transformation, quantile function, ordinary moments, and moment generating function. The method of maximum likelihood estimation is used to estimate the unknown parameters of the proposed distribution. Also, the performance of maximum likelihood estimators is investigated through Monte Carlo simulation study. The applicability of a new distribution is illustrated by the real data set.

**Keywords:** Exponential distribution, Beta generated family of distributions, Topp-Leone Distribution, Topp-Leone exponential distribution, T-X family

## Introduction

The exponential (E) distribution is applied to a wide range of fields: actuarial sciences, reliability, engineering, and others. Many researcher attempts to improve this distribution to become more flexible for modelling data such as, the generalized exponential or exponentiated exponential (Gupta and Kundu, 1999), the beta exponential (BE) (Nadarajah and Kotz, 2006), the beta generalized exponential (Barreto-Souza et al., 2010) and the Topp-Leone exponential (TLE) (Al-Shomrani et al., 2016) distributions. Our main focus in this paper is to introduce a new modification of exponential distribution using the  $T$ - $X$  family of distributions (Alzaatreh et al., 2013). Let  $T$  be a random variable of a generator distribution with probability distribution function (pdf)  $r(t)$  defined on  $[p, q]$  and let  $X$  be a parent random variable with cumulative distribution function (cdf)  $G(x)$ . The cdf of  $T$ - $X$  family is given by  $T$

$$F_{T-X}(x) = \int_p^{W(G(x))} r(t) dt, \quad (1)$$

where  $W(G(x))$  be a function of  $G(x)$  and satisfy the conditions as follows

1.  $W(G(x)) \in [p, q]$ ,
2.  $W(G(x))$  is differentiable and monotonically non-decreasing,
3.  $W(G(x)) \rightarrow p$  as  $x \rightarrow -\infty$  and  $W(G(x)) \rightarrow q$  as  $x \rightarrow \infty$ .

Eugene et al. (2002) pioneered this method, referred to as the beta generated (BG) family of distributions, by utilizing a beta random variable  $T$  via  $W(G(x)) = G(x)$ . Let  $G(x; \xi)$  be a parent cdf and let  $g(x; \xi) = dG(x; \xi)/dx$  be a parent probability density function (pdf) of a random variable  $X$  with parameters  $a, b$ , and the vector of parameters  $\xi$ . The pdf of BG family is

$$f_{BG}(x; a, b, \xi) = \frac{1}{B(a, b)} g(x; \xi) G(x; \xi)^{a-1} [1 - G(x; \xi)]^{b-1}, \quad a, b > 0, \quad (2)$$

where  $B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$  is the beta function. The cdf of BG family is

$$F_{BG}(x; a, b, \xi) = I_{G(x; \xi)}(a, b) \quad (3)$$

where the function  $I_{G(x; \xi)}(a, b)$  denotes the incomplete beta ratio function defined by

$$I_{G(x; \xi)}(a, b) = \frac{B_{G(x; \xi)}(a, b)}{B(a, b)}$$

where  $B_{G(x; \xi)}(a, b) = \int_0^{G(x; \xi)} t^{a-1} (1-t)^{b-1} dt$  is the incomplete beta function. Furthermore, by using this method, Al-Shomrani et al. (2016) introduced the Topp-Leone generated (TLG) family of distributions and the TLE distribution with its properties and application to the times to failure of components. Let  $X$  be a random variable having TLE distribution with parameters  $c, \lambda > 0$ , denoted  $X \sim \text{TLE}(c, \lambda)$ . The cdf and pdf of TLE distribution are given by

$$G_{\text{TLE}}(x; c, \lambda) = (1 - e^{-2\lambda x})^c, \quad x > 0 \quad (4)$$

and

$$g_{\text{TLE}}(x; c, \lambda) = 2c\lambda e^{-2\lambda x} (1 - e^{-2\lambda x})^{c-1}, \quad x > 0, \quad (5)$$

respectively. The rest of paper is structured as follows. In part of results, a new modification of exponential distribution called the beta Topp-Leone exponential distribution is proposed in Section 1. Some expansions of the beta Topp-Leone exponential distribution are obtained in Section 2. Some of its mathematical properties are investigated in Section 3. The proposed distribution parameters are estimated by maximum likelihood estimation in Section 4. A Monte Carlo simulation study is provided in Section 5. In Section 6, the flexibility of the proposed distribution will be explored through application to real data sets. Finally, the last section is the conclusion.

## Results

### 1. The Beta Topp-Leone Exponential Distribution

We introduce the beta Topp-Leone exponential (BTLE) distribution by setting  $G(x; \xi) = G_{\text{TLE}}(x; c, \lambda)$  in Equation (3). Let  $X$  be a random variable having BTLE distribution with the vector of parameters  $\Theta = (a, b, c, \lambda)^T$  where  $a, b, c, \lambda > 0$ , denoted  $X \sim \text{BTLE}(a, b, c, \lambda)$ . The cdf and pdf of the BTLE distribution are obtained as

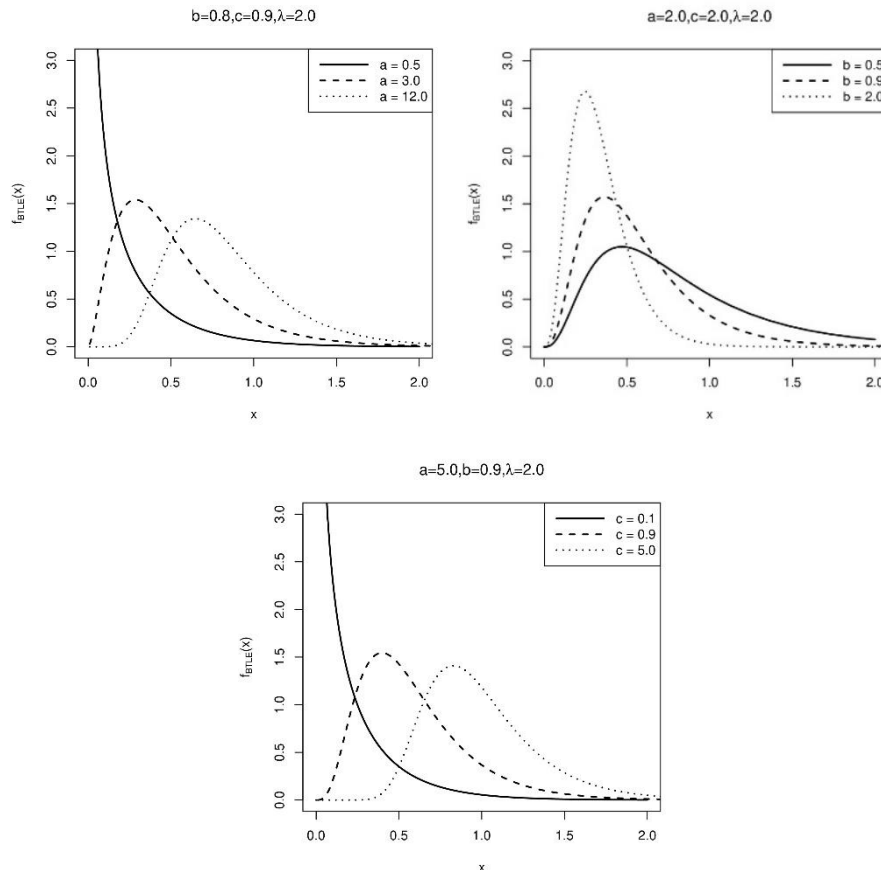
$$F_{\text{BTLE}}(x; \Theta) = I_{(1-e^{-2\lambda x})^c}(a, b), \quad x > 0, \quad (6)$$

and

$$f_{\text{BTLE}}(x; \Theta) = \frac{2c\lambda}{B(a, b)} e^{-2\lambda x} (1 - e^{-2\lambda x})^{ac-1} [1 - (1 - e^{-2\lambda x})^c]^{b-1}, \quad x > 0, \quad (7)$$

respectively. The BTLE distribution reduces to the TLE distribution when  $a = 1$  and  $b = 1$ . If  $c = 1$  it reduces to the beta transmuted exponential distribution with transmuted parameter equals to 1. If  $b = 1$  in addition to  $c = 1$ , it gives as the exponentiated transmuted exponential distribution with transmuted parameter equals to 1. The exponentiated Topp-Leone exponential distribution is also a sub-distribution when  $b = 1$ . The transmuted exponential distribution with transmuted parameter equals to 1 is clearly a sub-distribution for  $a = 1, b = 1$  and  $c = 1$ . Plots

of the BTLE pdf for some specific values of parameters  $a, b, c$ , and  $\lambda$  are shown in Figures 1, respectively.



**Figure 1** Plots of the BTLE pdf for some specific values of the parameters

## 2. Expansions for the BTLE Distribution

Some useful expansions for Equations (6) and (7) can be derived using the exponentiated exponential (EE) distribution. Let  $X_\alpha$  be a random variable having the EE distribution with parameters  $\alpha, \lambda > 0$ , denoted  $X_\alpha \sim EE(\alpha, \lambda)$  and the pdf and cdf of EE distribution are  $f_{EE}(x; \alpha, \lambda) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1}e^{-\lambda x}$  and  $F_{EE}(x; \alpha, \lambda) = (1 - e^{-\lambda x})^\alpha$ , respectively.

Firstly, for real non-integer  $b > 0$ , the term of  $(1 - t)^{b-1}$  under the integral is replaced by the power series, and is expressed as

$$\begin{aligned} \int_0^x t^{a-1}(1-t)^{b-1}dt &= \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} \int_0^x t^{a+i-1}dt \\ &= \sum_{i=0}^{\infty} \frac{1}{a+i} (-1)^i \binom{b-1}{i} x^{a+i}, \end{aligned}$$

where the binomial coefficient  $\binom{b-1}{i} = \Gamma(b)/\Gamma(b-i)!i!$  is defined for any real  $b$ . From Equation (6), we obtain

$$F_{BTLE}(x; \theta) = \frac{1}{B(a, b)} \sum_{i=0}^{\infty} \frac{1}{a+i} (-1)^i \binom{b-1}{i} \left[ 1 - \left( 1 - (1 - e^{-\lambda x}) \right)^2 \right]^{c(a+i)}.$$

Using the binomial expansion once more, the cdf of BTLE family will be

$$F_{BTLE}(x; \theta) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{2j} w_{i,j,k} F_{EE}(x; k, \lambda) \quad (8)$$

where,

$$w_{i,j,k} = \frac{(-1)^{i+j+k}}{B(a, b)(a+i)} \binom{b-1}{i} \binom{c(a+i)}{j} \binom{2j}{k}.$$

By differentiating Equation (8), we obtain

$$f_{BTLE}(x; \theta) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{2j} w_{i,j,k} f_{EE}(x; k, \lambda). \quad (9)$$

If  $b > 0$  is an integer, the index  $i$  in Equations (8) and (9) will stop at  $b-1$ , and if both  $a$  and  $c$  are integers, then the index  $j$  will run and stop at  $c(a+i)$ .

### 3. Mathematical Properties

In this section, some mathematical properties of the BTLE distribution, including transformation, quantile function, ordinary moments and moment generating function, are provided.

#### 3.1 Transformation

Let  $B$  be a random variable having a beta distribution with parameters  $a$  and  $b$ . By inverting Equation (4), we will obtain

$$X = \left\lceil \frac{\log(1 - B^{1/c})}{-2\lambda} \right\rceil \quad (10)$$

follows a BTLE distribution with parameters  $a$ ,  $b$ ,  $c$ , and  $\lambda$ .

#### 3.2 Quantile function

Let  $U$  be a random variable having a uniform on interval  $(0,1)$ . By inverting Equation (6), the quantile function,  $Q(u) = F^{-1}(u)$ , of the BTLE distribution is

$$Q_{BTLE}(u; \theta) = \left\lceil \frac{\log \left[ 1 - \left( I_u^{-1}(a, b) \right)^{1/c} \right]}{-2\lambda} \right\rceil, \quad 0 < u < 1, \quad (11)$$

where  $I_u^{-1}(a, b)$  is the inverse of the incomplete beta ratio function (Majumder and Bhattacharjee, 1973).

#### 3.3 Ordinary Moments

The  $r$ th ordinary moment of the BTLE distribution can be straightforwardly obtained from Equation (9) as

$$\mu'_r = E(X^r) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{2j} w_{i,j,k} E(X_k^r). \quad (12)$$

where  $E(X_k^r)$  is the  $r$ th ordinary moment of the EE distribution. Nadarajah (2011) that the  $r$ th moment of the EE distribution is

$$E[X_\alpha^r] = \frac{(-1)^r \alpha}{\lambda^r} \frac{\partial^r}{\partial d^r} B(\alpha, d+1-\alpha) \Big|_{d=\alpha}. \quad (13)$$

From Equations (12) and (13), the  $r$ th ordinary moment of the BTLE distribution is

$$\mu'_r = \frac{(-1)^r}{\lambda^r} \sum_{i,j=0}^{\infty} \sum_{k=0}^{2j} k w_{i,j,k} \frac{\partial^r}{\partial d^r} B(k, d+1-k) \Big|_{d=k}. \quad (14)$$

### 3.4 Moment Generating Function

The moment generating function (mgf) of  $X$ ,  $M_X(t) = E(e^{tX})$ , can be written from Equation (9) as

$$M_X(t) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{2j} w_{i,j,k} M_{X_k}(t), \quad (15)$$

where  $M_{X_k}(t)$  is the mgf of the EE distribution. Nadarajah (2011) shows that the mgf of the EE distribution is

$$M_{X_\alpha}(t) = \alpha B\left(1 - \frac{t}{\lambda}, \alpha\right). \quad (16)$$

From Equations (15) and (16), the mgf of the BTLE distribution is

$$M_X(t) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{2j} k w_{i,j,k} B\left(1 - \frac{t}{\lambda}, k\right). \quad (17)$$

## 4. Maximum Likelihood Estimation

Let  $\mathbf{x} = (x_1, \dots, x_n)^T$  be a random sample of size  $n$  from the BTLE distribution. The likelihood function for the vector of parameters  $\boldsymbol{\theta} = (a, b, c, \lambda)^T$  is

$$\begin{aligned} L(\boldsymbol{\theta}; \mathbf{x}) &= \prod_{i=1}^n f_{\text{BTLE}}(x_i; \boldsymbol{\theta}) \\ &= \prod_{i=1}^n \left[ \frac{2c\lambda}{B(a,b)} e^{-2\lambda x_i} (1 - e^{-2\lambda x_i})^{ac-1} [1 - (1 - e^{-2\lambda x_i})^c]^{b-1} \right]. \end{aligned}$$

The corresponding log-likelihood function is

$$\begin{aligned} \ell(\boldsymbol{\theta}; \mathbf{x}) &= \log L(\boldsymbol{\theta}; \mathbf{x}) = \sum_{i=1}^n \log f_{\text{BTLE}}(x_i; \boldsymbol{\theta}) \\ &= -n \log B(a,b) + n \log(2) + n \log(c) + n \log(\lambda) + n \log(c) - 2\lambda \sum_{i=1}^n x_i^\lambda \\ &\quad + (ac-1) \sum_{i=1}^n \log(1 - v_i^2) + (b-1) \sum_{i=1}^n \log[1 - (1 - v_i^2)^c], \end{aligned}$$

where  $v_i = e^{-\lambda x_i}$  is a transformed observation. The maximum likelihood estimator (MLE)  $\hat{\Theta}$  of the vector of unknown parameters  $\Theta$  in Equation (18) can be obtained by the score function

$$U(\Theta) = \frac{\partial \ell(\Theta; \mathbf{x})}{\partial \Theta} = 0.$$

The components of the score function are

$$U_a(\Theta) = n(\psi(a+b) - \psi(a)) + c \sum_{i=1}^n \log(1 - v_i^2),$$

$$U_b(\Theta) = n(\psi(a+b) - \psi(b)) + \sum_{i=1}^n \log[1 - (1 - v_i^2)^c],$$

$$U_c(\Theta) = \frac{n}{c} + a \sum_{i=1}^n \log(1 - v_i^2) - (b-1) \sum_{i=1}^n \frac{(1 - v_i^2)^c \log(1 - v_i^2)}{1 - (1 - v_i^2)^c},$$

$$U_\lambda(\Theta) = \frac{n}{\lambda} - 2 \sum_{i=1}^n x_i + 2(ac-1) \sum_{i=1}^n \frac{x_i}{v_i^{-2} - 1} - 2c(b-1) \sum_{i=1}^n \frac{x_i(1 - v_i^2)^c}{(v_i^{-2} - 1)(1 - (1 - v_i^2)^c)},$$

where  $v_i = e^{-\lambda x_i}$  and  $\psi(\cdot)$  is the digamma function. However, these non-linear equations cannot be solved analytically. Therefore, the value of MLE  $\hat{\Theta}$  that maximizes the log-likelihood function can be computed numerically from the non-linear equations utilizing the optimr package (Nash, 2016) in R programming language (R Core Team, 2020).

For interval estimation on the vector of parameters  $\Theta$ , the observed Fisher information matrix  $J(\Theta)$  is obtained because it is not always possible to compute expected Fisher information matrix  $I(\Theta)$ . The  $4 \times 4$  observed Fisher information matrix is defined by

$$J(\Theta) = -\frac{\partial^2 \ell(\Theta; \mathbf{x})}{\partial \Theta \partial \Theta^T} = - \begin{bmatrix} J_{aa} & J_{ab} & J_{ac} & J_{a\lambda} \\ \cdot & J_{bb} & J_{bc} & J_{b\lambda} \\ \cdot & \cdot & J_{cc} & J_{c\lambda} \\ \cdot & \cdot & \cdot & J_{\lambda\lambda} \end{bmatrix}$$

where the elements of  $J(\Theta)$  are given by

$$J_{aa} = n(\psi'(a+b) - \psi'(a)),$$

$$J_{bb} = n(\psi'(a+b) - \psi'(b)),$$

$$J_{cc} = -\frac{n}{c^2} - (b-1) \sum_{i=1}^n \frac{(1 - v_i^2)^c \log^2(1 - v_i^2)}{[1 - (1 - v_i^2)^c]^2},$$

$$J_{\lambda\lambda} = -\frac{n}{\lambda^2} - 4(ac-1) \sum_{i=1}^n \frac{x_i^2 v_i^{-2}}{(v_i^{-2} - 1)^2} - 4c(b-1) \sum_{i=1}^n \frac{x_i^2 (1 - v_i^2)^c \{v_i^{-2} [(1 - v_i^2)^c - 1] + c\}}{(v_i^{-2} - 1)^2 [1 - (1 - v_i^2)^c]^2},$$

$$J_{ab} = n\psi'(a+b),$$

$$J_{ac} = \sum_{i=1}^n \log(1 - v_i^2),$$

$$J_{a\lambda} = 2c \sum_{i=1}^n \frac{x_i}{v_i^{-2} - 1},$$

$$J_{bc} = \sum_{i=1}^n \frac{(1 - v_i^2)^c \log(1 - v_i^2)}{(1 - v_i^2)^c - 1},$$

$$J_{b\lambda} = 2c \sum_{i=1}^n \frac{x_i (1 - v_i^2)^c}{(v_i^{-2} - 1)[(1 - v_i^2)^c - 1]},$$

$$J_{c\lambda} = 2a \sum_{i=1}^n \frac{x_i}{v_i^{-2} - 1} + 2(b-1) \sum_{i=1}^n \frac{x_i(1-v_i^2)^c[(1-v_i^2)^c - c \log(1-v_i^2) - 1]}{(v_i^{-2} - 1)[1 - (1-v_i^2)^c]^2},$$

where  $v_i = e^{-\lambda x_i}$  and  $\psi'(\cdot)$  is the derivative of the digamma function. The total Fisher information matrix is  $J_n(\boldsymbol{\theta}) = nJ(\boldsymbol{\theta})$ . In addition, the second partial derivatives can be numerically computed by using numDeriv package (Gilbert and Varadhan, 2019) in R programming language (R Core Team, 2020).

The asymptotic distribution of  $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$  is multivariate normal  $N_4(\mathbf{0}, I(\boldsymbol{\theta})^{-1})$  where  $I(\boldsymbol{\theta})$  is the expected Fisher information matrix. For construct asymptotic confidence intervals, the  $I(\boldsymbol{\theta})$  can be replaced by  $J(\hat{\boldsymbol{\theta}})$  that is the observed Fisher information matrix calculated at  $\hat{\boldsymbol{\theta}}$ . The corresponding asymptotic confidence intervals with significance level  $\alpha$  for each parameter are

$$\hat{a} \pm z_{\alpha/2} \sqrt{J^{aa}}, \quad \hat{b} \pm z_{\alpha/2} \sqrt{J^{bb}}, \quad \hat{c} \pm z_{\alpha/2} \sqrt{J^{cc}}, \quad \hat{\lambda} \pm z_{\alpha/2} \sqrt{J^{\lambda\lambda}},$$

where  $z_{\alpha/2}$  is the quantile  $1 - \alpha/2$  of the standard normal distribution and  $J^{aa}, J^{bb}, J^{cc}$ , and  $J^{\lambda\lambda}$  are the diagonal elements of variance-covariance matrix  $\boldsymbol{\Sigma} = J_n(\hat{\boldsymbol{\theta}})^{-1}$ .

## 5. Simulation study

A Monte Carlo simulation study is conducted to investigate the performance of the MLEs based on bias and root mean square error (RMSE). We consider sample sizes  $n = 15, 25, 50, 100, 250, 500$  and the different values of the BTLE parameters  $a, b, c$ , and  $\lambda$ :  $a = 0.6, b = 0.4, c = 1$ , and  $\lambda = 2$  (non-increasing pdf) and  $a = 2, b = 0.75, c = 1.5$ , and  $\lambda = 3$  (right-skewed pdf). The experiment is repeated 2000 times. An algorithm for generating a BTLE random variable  $X$  with parameters  $a, b, c$ , and  $\lambda$ :

- i. Generate a beta random variable with parameters  $a$  and  $b$ ,  $B \sim \text{Beta}(a, b)$ .
- ii. Set  $X = \left\lceil \frac{\log(1-B^{1/c})}{-2\lambda} \right\rceil$ .

Table 1 gives the average parameter estimates, average bias, and average RMSE of the MLEs. The results show that the MLEs are the asymptotically unbiased and consistent, i.e., the bias and RMSE decrease when the sample size increases.

**Table 1** The average parameter estimates, average bias, and average RMSE

Sample size	Parameters	BTLE(0.6,0.4,1,2)			BTLE(2,0.75,1.5,3)		
		Parameter estimates	Bias	RMSE	Parameter estimates	Bias	RMSE
15	$a$	2.662	2.062	4.366	5.125	3.125	6.750
	$b$	4.608	4.208	11.160	10.186	9.436	18.345
	$c$	4.039	3.039	7.682	7.294	5.794	10.848
	$\lambda$	4.295	2.295	4.827	5.284	2.284	6.609
25	$a$	2.561	1.961	3.990	3.039	3.125	6.750
	$b$	3.033	2.633	7.124	7.865	9.436	18.345
	$c$	2.549	1.549	4.844	7.670	6.170	12.594
	$\lambda$	4.083	2.083	4.278	5.122	2.122	5.689
50	$a$	1.811	1.211	2.619	3.056	1.056	3.019
	$b$	1.563	1.163	3.386	3.584	2.834	6.504
	$c$	1.673	0.673	2.472	3.799	2.299	6.086

100	$\lambda$	3.765	1.765	3.441	4.324	1.324	4.334
	$a$	1.469	0.869	1.849	2.497	0.497	2.317
	$b$	1.291	0.891	2.198	2.268	1.518	3.374
	$c$	1.286	0.286	1.356	3.421	1.921	4.932
250	$\lambda$	2.973	0.973	2.533	3.560	0.560	2.960
	$a$	1.205	0.605	1.382	2.861	0.861	2.321
	$b$	0.819	0.419	1.359	1.674	0.924	2.327
	$c$	1.154	0.154	1.003	2.161	0.661	2.061
500	$\lambda$	2.640	0.640	1.846	3.061	0.061	1.786
	$a$	0.852	0.252	0.896	2.440	0.440	1.911
	$b$	0.994	0.594	1.458	1.271	0.521	1.707
	$c$	1.293	0.293	1.106	2.247	0.747	2.044
	$\lambda$	2.176	0.176	1.511	2.945	-0.055	1.271

## 6. Application

In this section, the fitted results of the BTLE, BE, TLE, and E distributions are compared with real data set to demonstrate the flexibility and applicability of the proposed distribution among the other distributions. In order to evaluate whether the distribution is appropriate, the many statistical tools are considered: the criteria of Akaike's information criterion (AIC) (Akaike, 1974), Bayesian information criterion (BIC) (Schwarz, 1978) and Anderson-Darling test,  $A^*$  (Stephens, 1974). The data set is the strength of glass fibres of lengths 1.5 cm from the National Physical Laboratory in England (Smith and Naylor, 1987). Table 2 gives a descriptive statistics summary of these data.

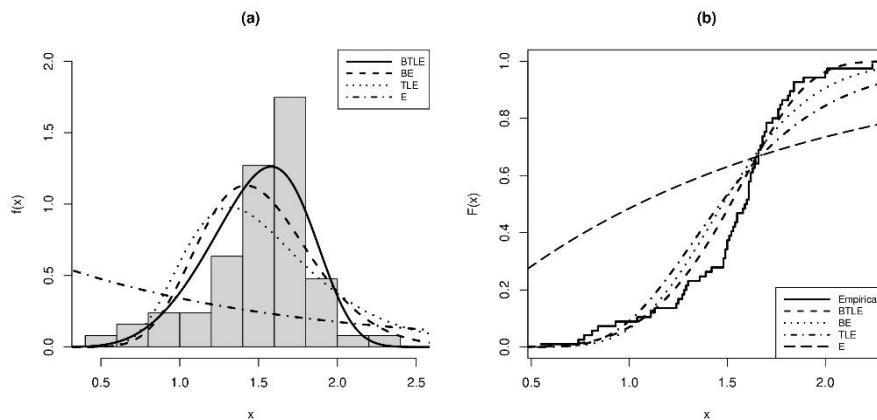
**Table 2** Descriptive statistics summary of the data

n	Mean	Standard Deviation	Median	Min	Max	Skewness	Kurtosis
63	1.51	0.32	1.59	0.55	2.24	-0.88	0.80

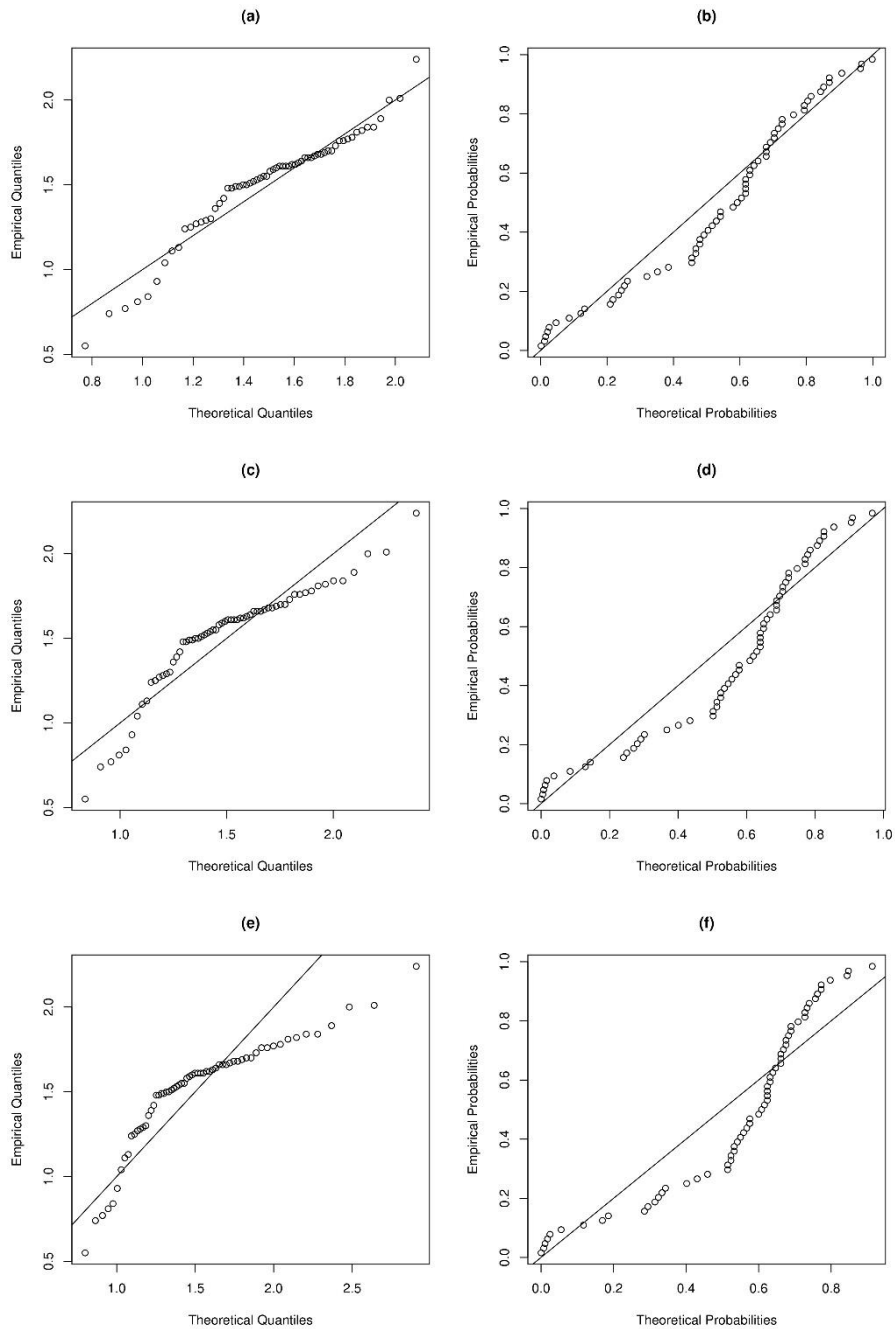
The maximum likelihood estimates (the standard error (SE) is given in parentheses), the values of minus loglikelihood, AIC, BIC and  $A^*$  of the distributions for real data set are shown in Table 3. The lower the values of these statistics indicate a better fit to the data. Since these findings suggest that the BTLE distribution has the lowest AIC, BIC and  $A^*$  values, it follows that the BTLE distribution could be a suitable distribution for the fitting of the data. The histogram and the estimated pdf plots of the data are illustrated in Figure 2(a). In Figure 2(b), the empirical cdf and the estimated cdf plots for the data are shown. Furthermore, the goodness-of-fit plots for BTLE, BE, TLE, and E distributions that consist of Q-Q and P-P plots are presented in Figure 3, respectively. The conclusion of these plots indicates that BTLE distribution provides a better fit for the data.

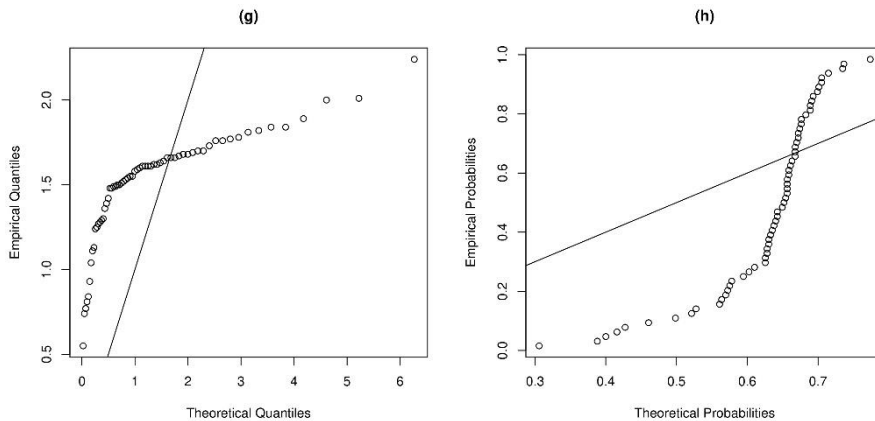
**Table 3** The MLEs (SEs), minus log-likelihood, AIC, BIC, and Anderson-Darling test

Parameter estimates	Distributions			
	E	TLE	BE	BTLE
$\hat{a}$	-	-	17.476 (3.127)	0.381 (0.195)
$\hat{b}$	-	-	47.795 (115.945)	115.368 (130.168)
$\hat{c}$	-	31.296 (9.497)	-	23.309 (15.724)
$\hat{\lambda}$	0.664 (0.084)	1.305 (0.119)	0.209 (0.443)	0.454 (0.178)
-log-likelihood	88.83	31.383	24.002	15.467
AIC	179.661	66.767	54.004	38.934
BIC	181.804	71.053	60.434	47.506
$A^*$	3.127	4.286	3.127	1.378
(p-value)	(0.024)	(0.006)	(0.024)	(0.208)



**Figure 2** (a) The histogram of the data and plots estimated densities of the fitted distributions  
(b) The empirical cdf of the data and plots estimated cdfs of the fitted distributions





**Figure 3** (a)-(b) Q-Q and P-P plots for the BTLE distribution, (c)-(d) Q-Q and P-P plots for BE distribution, (e)-(f) Q-Q and P-P plots for TLE distribution, and (g)-(h) Q-Q and P-P plots for E distribution fitted to the data

In addition, the variance-covariance matrix for the data is

$$\Sigma = \begin{bmatrix} 0.038 & 5.100 & -2.826 & -0.027 \\ 5.100 & 16937.844 & -882.456 & -15.980 \\ -2.826 & -882.456 & 247.195 & 2.643 \\ -0.027 & -15.980 & 2.643 & 0.032 \end{bmatrix}$$

Finally, we will obtain the  $100(1 - \alpha)\%$  asymptotic confidence intervals of the BTLE parameters  $\Theta = (a, b, c, \lambda)^T$ , where  $\alpha = 0.90, 0.95, 0.99$  in Table 4.

**Table 4** The asymptotic confidence intervals of the BTLE parameters based on the data

Parameters	90%		95%		99%	
	Lower	Upper	Lower	Upper	Lower	Upper
$a$	0.060	0.702	-0.001	0.763	-0.121	0.883
$b$	-98.739	329.475	-139.757	370.493	-219.923	450.659
$c$	-2.555	49.173	-7.509	54.127	-17.193	63.811
$\lambda$	0.161	0.747	0.105	0.803	-0.004	0.921

## Conclusions

A new four-parameter distribution called the beta Topp-Leone exponential distribution is proposed. Its cdf and pdf of the proposed distribution are derived. Some of its mathematical properties, i.e. some expansions of pdf and cdf, transformation, quantile function, ordinary moments and mgf are presented. The maximum likelihood estimation is used to find out the parameter estimates for the BTLE distribution. Through Monte Carlo simulation study we demonstrated that, the bias and RMSE of MLEs decrease as the sample size increases. Results of fitting the BTLE, BE, TLE and E distributions to the real data set are evaluated. Considering the values of AIC, BIC, and Anderson-Darling test, it suggests that the BTLE distribution could

outperform the other distributions. We hope that the BTLE distribution may attract wider applications in various areas such as reliability, engineering, and others.

## Acknowledgement

This paper was made possible by Department of Statistics, Faculty of Science, Kasetsart University, Graduate School Kasetsart University and Science Achievement Scholarship of Thailand (SAST).

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