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Approximation Algorithm for Multi-Product Newsboy Problem with Stochastic Yield

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ABSTRACT

In this paper, we consider the newsboy problem when two uncertainties are presented, including demand and yield. Based on previous research in this arena, the optimal solution can be obtained analytically for uniform distribution. However, for other distributions the problem becomes much more complex and difficult, in the other words no optimal solution can be found. The reason of this dilemma could be non-integrable property of the objective function. Therefore, we extend that of Karl in 2004 to present the approximation algorithm for general probability distribution, both demand and yield's randomness are incorporated by different types of probability distribution such as Normal, Uniform and Beta distributions. The algorithm framework consists of two measures: one involving the approximation of the first derivative of the objective functions based on numerical integration; and second involving the problem solving steps based on Newton's method. From the experiment, it can give the good solution for Normal and other distributions yet it is easy to implement by insert function in available spreadsheet program.

Keywords: Newsboy problem, Stochastic yield, Perishable products, Inventory

1. INTRODUCTION

The classical newsboy problem is designed for perishable products, which can be carried in the inventory for only a period of time before it can no longer be sold. Various types of perishable products are for example newspaper and magazine, flowers, fresh food, fresh vegetable, etc. These products have different shelf life and to dispose them need additional cost. Not only cost of disposing but also there are two remaining cost components still required. The first one is inventory holding cost which is incurred when a seller orders more than he can sell. The second one is loss of opportunity cost which is incurred when he orders less than can be sold. When all cost components are combined, they play an important role in business cost that need to be minimized for surviving in today's high competitive environment. Therefore, it is important to find the right order quantity match to customer demands and give the maximum profit or the minimum cost.

The newsboy problem has been increasingly interested as illustrated by several papers published since 1988 and covered various extensions. Khouja [10] classified the newsboy problem into 11 categories: extensions to different objectives and utility functions; extensions to different supplier pricing policies; extensions to different pricing policies and discount structures; extensions to random yield; extensions to different states of information about demand; extensions to constrained multi-product; extensions to multi-product with substitution; extensions to multi-echelon systems; extensions to multi-location models; extensions to models with more than one period; and other extensions. However, the literature in the random yield avenue has rarely found when compares with others.

Random yield is occurred when the order quantity contained defective product rather than perfect quality as consider in traditional newsboy problem. This leads to the reduction of quantities of end product. The random yield is a very important problem and generally realized in the real world applications. The random yield can be characterized in many production systems such as production of agriculture (i.e. production of fruits or vegetable), production of computer processors and production of chemicals, and etc. For example, in semiconductor manufacturing, it is possible to face yield losses which can exceed 80%, as mentioned by Nahmias [14]. In fast growing

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remanufacturing industry, the disassembly processes face with high yield fluctuation stemmed from the limited knowledge of the quality of used products. Others than production system, in procurement processes, we still can face the yield uncertainty derived from unreliable delivery quantities of suppliers. As one can see, the random yield is a very important problem. Despite how important of the problem, insufficient attention has been given to this problem when considering multi items with limited resources, Abdel-Malek et al. [2].

The classical newsboy problem was originally developed by Hadley and Whitin. There are several related articles and among these the most comprehensive articles can be found in Lau and Lau [11], [12]. They developed a simple algorithm for solving multi-product constrained newsboy model. They also mentioned that the obtained solution might be negative order quantity when budget constrain is tight. Ben-Daya and Raouf [4] and Erlebacher [5] introduced optimal and heuristic solution method for the newsboy problem with one constraint. Niederhoff [15] introduced a linear programming formulation for the multi-constraint newsvendor problem where the objective function is approximated and optimized by linear segments. Abdel-Malek and Areeratchakul [1] developed a quadratic programming approach for solving the multi-constraint problem and utilizes familiar software packages such as Excel to solve the problem.

For the random yield case, several articles focus on the stochastically proportional yield which assumes that the fraction of good units is a random variable independent of the batch size. This type of yield is suitable for the production environment where batch sizes are relatively large. For more details, interested readers can refer to Gerchak et al. [7] and Henig and Gerchak [8] where they proved that the optimal order quantity will not be a linear function of the inventory level. Ehrhardt and Taube [6] derived the closed-form solutions for the problem when demand and yield distributions are uniform distribution. Lee and Yano [13] provided a comprehensive review of the lot sizing problem with random yields. Noori and Keller [16] derived the optimal solution order quantity for the unconstrained newsboy problem with random yield for both random demand and yield. Inderfurth [9] derived the optimal solution for single item newsboy problem for uniformly distributed demand and yield. Abdel-Malek et al. [3] extended their works from that of Abdel-Malek and Montanari [2] to cover the random yield case. They assumed that the decision variable (the amount to be planted) is the upper bound of yield and the starting inventory is zero. They developed an algorithm based on iterative process to solve the multi-item constrained problem for general distributed demand and yield.

From the literature reviews, one can see that the previous works mainly focuses on single item uncapacitated random yield case and insufficient attention on multi-item with capacitated case. In this paper, we focus on this limitation by developing a methodology for solving the problem for general demand and yield distributions cases. Our methodology are based on two main steps, which are the approximation of the objective function by using composite trapezoid numerical integration method, and iterative solution finding steps using Lagrange multiplier and Newton's method. Additionally, we also provide a numerical example to illustrate the application of the model. This paper is organized as follows. The paper begins with an introduction and literature review in section 1. Section 2 and 3 present the model and its necessary preliminaries. Section 4 illustrates the numerical example for different demand distribution as well as yield distribution. Finally, we present the conclusions in Section 5.

2. MODEL FORMULATION

In this section we describe the newsboy problem with stochastic demand and stochastically proportional yield. The model is well established and mentioned in Ehrhardt and Taube [6], and Inderfurth [9]. The model assumes that lead time is zero and cost components are strictly proportional related to production and inventory. In this paper we extend their works to cover more than one product type and limited available resources.

To be more specific, let consider a fruit retailers in a market, they must find out how much they have to order (x_τ) for each type of fruit (τ) within available resources *(Bg)*. Each fruit has demand (D_τ) and unit cost (c_τ). If they order more than customer demand, the overage cost incurs with inventory unit holding cost (h_τ). In contrast, if they order less than its demand, the underage cost incurs with inventory shortage cost per unit (v_τ). Moreover, ordered fruits can be damaged from several random factors such as collision during transportation, infection, unsuitable storage conditions, etc. If the fruit retailers receive the order in full quantity, the yield is 100%, but if their orders are fulfilled partially the yield is less than 100%. However, yield can never be negative. The demand and yield realizations are assumed to have upper bound D_τ^+ and Y_τ^+ and lower bound at zero, shown as $0 \le D_\tau \le D_\tau^+$ and $0 \le Y_\tau \le Y_\tau^+$. The expected total cost function can be formulated as

$$TC = \sum_{\tau=1}^{N}\{PC_\tau + UC_\tau + OC_\tau\}, \qquad (2.1)$$

where PC_τ, UC_τ, and OC_τ are purchasing cost, expected underage cost, and expected overage cost, respectively denoted as follows.

$$PC_\tau(x_\tau) = c_\tau x_\tau \qquad (2.2)$$

$$UC_\tau(x_\tau) = \int_0^\infty \int_{I_\tau+Y_\tau x_\tau}^\infty v_\tau(D_\tau - I_\tau - Y_\tau x_\tau)\, f(D_\tau)\, f(Y_\tau)\, dD_\tau dY_\tau \qquad (2.3)$$

$$OC_\tau(x_\tau) = \int_0^\infty \int_0^{I_\tau+Y_\tau x_\tau} h_\tau(I_\tau + Y_\tau x_\tau - D_\tau)\, f(D_\tau)\, f(Y_\tau)\, dD_\tau dY_\tau \qquad (2.4)$$

Since resources such as available budgets or shelf spaces should not be freely available and order quantity should never be negative. Therefore, we incorporate their limited resources and non-negative constraint as below.

$$\sum_{\tau=1}^{N} c_\tau x_\tau \le Bg \qquad (2.5)$$

$$x_\tau \ge 0, \text{ for } \tau = 1 \dots N$$

Before we go further in formulation detail, let summarize all notations used in this paper defined in Table 1:

Table 1 Notations Table

Notations	Definition
TC	Expected total cost
PC_τ	Expected purchasing cost
UC_τ	Expected underage cost
OC_τ	Expected overage cost
τ	Product type
N	Total number of product type
c_τ	Unit cost for purchasing
x_τ	Order quantity
I_τ	Starting inventory
Y_τ	Stochastic yield fraction
h_τ	Unit holding cost
v_τ	Unit shortage cost
D_τ	Stochastic demand

Table 2 (continue) Notations Table

Notations	Definition
TC	Expected total cost
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c_τ	Unit cost for purchasing
x_τ	Order quantity
I_τ	Starting inventory
Y_τ	Stochastic yield fraction
h_τ	Unit holding cost
v_τ	Unit shortage cost
D_τ	Stochastic demand
Bg	Available resources
$f(D_\tau)$	Probability density function of demand
$f(Y_\tau)$	Probability density function of yield
$F(D_\tau)$	Cumulative distribution function of demand
$F(Y_\tau)$	Cumulative distribution function of yield
D_τ^+	Upper bound of demand
Y_τ^+	Upper bound of yield

Related to the specified upper bounds and lower bounds, two scenarios have to be distinguished for formulating overage and underage cost properly: 1) when $I_\tau + Y_\tau^+ x_\tau \le D_\tau^+$; and 2) when $I_\tau + Y_\tau^+ x_\tau > D_\tau^+$

The 1st scenario:

The first scenario, the underage and overage inventories are incurred due to demand randomness. We can formulate the expected total cost as

$$TC_1(x_\tau) = \sum_{\tau=1}^{N} \left\{ \begin{array}{c} c_\tau x_\tau + \\ \int_0^{Y_\tau^+} \int_0^{I_\tau+Y_\tau x_\tau} h_\tau(I_\tau + Y_\tau x_\tau - D_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau + \\ \int_0^{Y_\tau^+} \int_{I_\tau+Y_\tau x_\tau}^{D_\tau^+} v_\tau(D_\tau - I_\tau - Y_\tau x_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau \end{array} \right\} \qquad (2.6)$$

To consider the available budget into the objective function, we add Lagrange multiplier (λ) in with the Lagrange function. It is defined as:

$$TC_1(x_\tau, \lambda) = \sum_{\tau=1}^{N} \left\{ \begin{array}{c} c_\tau x_\tau + \\ \int_0^{Y_\tau^+} \int_0^{I_\tau+Y_\tau x_\tau} h_\tau(I_\tau + Y_\tau x_\tau - D_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau + \\ \int_0^{Y_\tau^+} \int_{I_\tau+Y_\tau x_\tau}^{D_\tau^+} v_\tau(D_\tau - I_\tau - Y_\tau x_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau \end{array} \right\} + \lambda\left(\sum_{\tau=1}^{N} c_\tau x_\tau - Bg\right) \qquad (2.7)$$

Taking the first derivative of expected total cost, we obtain:

$$\frac{\partial TC_1(x_\tau)}{\partial x_\tau} = g_1(x_\tau, \lambda) = (h_\tau + v_\tau) \int_0^{Y_\tau^+} Y_\tau f(Y_\tau) F_{D_\tau}(I_\tau + Y_\tau x_\tau) dY_\tau - v_\tau \mu_{Y_\tau} + c_\tau + \lambda c_\tau \qquad (2.8)$$

The second derivative of expected total cost can be defined as:

$$\frac{\partial^2 TC_1(x_\tau)}{\partial x_\tau^2} = g_1^{'}(x_\tau,\lambda) = (h_\tau + v_\tau)\int_0^{Y_\tau^+} Y_\tau^2 f(Y_\tau) f_{D_\tau}(I_\tau + Y_\tau x_\tau)dY_\tau \tag{2.9}$$

The 2nd scenario:

The second scenario, the overage inventory is incurred certainly due to yield realization. We can formulate the expected total cost as:

$$TC_2(x_\tau) = \sum_{\tau=1}^{N}\left\{\begin{array}{c} c_\tau x_\tau + \int_0^{\frac{D_\tau^+ - I_\tau}{x_\tau}} \int_0^{I_\tau + Y_\tau x_\tau} h_\tau(I_\tau + Y_\tau x_\tau - D_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau + \\ \int_{\frac{D_\tau^+ - I_\tau}{x_\tau}}^{Y_\tau^+} \int_0^{D_\tau^+} h_\tau(I_\tau + Y_\tau x_\tau - D_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau + \\ \int_0^{\frac{D_\tau^+ - I_\tau}{x_\tau}} \int_{I_\tau + Y_\tau x_\tau}^{D_\tau^+} v_\tau(D_\tau - I_\tau - Y_\tau x_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau \end{array}\right\} \tag{2.10}$$

With similar fashion as described in the 1st scenario by adding Lagrange multiplier(λ), taking first and second derivatives, therefore summarize of these equations are shown in Eq. 2.11-2.13, respectively:

$$TC_2(x_\tau) = \sum_{\tau=1}^{N}\left\{\begin{array}{c} c_\tau x_\tau + \int_0^{\frac{D_\tau^+ - I_\tau}{x_\tau}} \int_0^{I_\tau + Y_\tau x_\tau} h_\tau(I_\tau + Y_\tau x_\tau - D_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau + \\ \int_{\frac{D_\tau^+ - I_\tau}{x_\tau}}^{Y_\tau^+} \int_0^{D_\tau^+} h_\tau(I_\tau + Y_\tau x_\tau - D_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau + \\ \int_0^{\frac{D_\tau^+ - I_\tau}{x_\tau}} \int_{I_\tau + Y_\tau x_\tau}^{D_\tau^+} v_\tau(D_\tau - I_\tau - Y_\tau x_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau \end{array}\right\} + \lambda\left(\sum_{\tau=1}^{N} c_\tau x_\tau - Bg\right) \tag{2.11}$$

$$\frac{\partial TC_2(x_\tau)}{\partial x_\tau} = g_2(x_\tau,\lambda) = (h_\tau + v_\tau)\left[\begin{array}{c} \int_0^{\frac{D_\tau^+ - I_\tau}{x_\tau}} Y_\tau f(Y_\tau) F(D_\tau)(I_\tau + Y_\tau x_\tau)dY_\tau + \\ \int_{\frac{D_\tau^+ - I_\tau}{x_\tau}}^{Y_\tau^+} Y_\tau f(Y_\tau) dY_\tau \end{array}\right] - v_\tau \mu_\tau + c_\tau + \lambda c_\tau \tag{2.12}$$

$$\frac{\partial^2 TC_2(x_\tau)}{\partial x_\tau^2} = g_2^{'}(x_\tau,\lambda) = (h_\tau + v_\tau)\left[\int_0^{\frac{D_\tau^+ - I_\tau}{x_\tau}} Y_\tau^2 f(Y_\tau) f(D_\tau)(I_\tau + Y_\tau x_\tau)dY_\tau\right] \tag{2.13}$$

The next section describes the approximation algorithm for solving problem. We implement our algorithm with a numerical example when demand and yield are varied from Normal, Uniform, and Beta distributions.

3. THE APPROXIMATION ALGORITHM

From the aforementioned section, one can see that the model is complex and not easy to solve. An analytical solution can only be given for specific type of demand and yield distributions such as Uniform or Exponential distributions. However, in practical applications the problem covers more than just only these two distributions and faces with several production constraints. This leads to necessity in developing supported algorithm. The algorithm is developed based on two main steps: (1) approximation of first derivative of the expected total cost using numerical integration technique; and (2) iterative processes for finding problem solutions using Newton's method.

3.1 Approximation of the first and second derivatives of the expected total cost function

The approximation is divided into two subsections for the first scenario and the second scenario, respectively.

3.1.1 Approximation for the first scenario

Since the analytical solution cannot obtain for the case of general distribution, the first order derivative function can be approximated using composite trapezoid numerical integration method:

$$\int_{a_j}^{b_j} f(t)dt = \frac{\gamma_j}{2}\left[f(a_j) + 2f(\theta_{1,j}) + 2f(\theta_{2,j}) + f(b_j)\right] \tag{3.1}$$

for $i = 1, 2$ and $j = 1, 2, 3$

when $\gamma_j = \frac{(b_j - a_j)}{3}, \theta_{i,j} = a_j + \gamma_j i$

and a_j represents lower bound of integration, b_j represents upper bound of integration

Hence, let $(a_1 = 0, b_1 = Y_\tau^+)$, then we calculate $\gamma_1, \theta_{1,1}, \theta_{2,1}$ as

$$\gamma_1, \theta_{1,1} = \frac{(b_1 - a_1)}{3} = \frac{(Y_\tau^+)}{3}, \theta_{2,1} = a_1 + 2\gamma_1 = \frac{2(Y_\tau^+)}{3}$$

From equation 2.8, the term $\int_0^{Y_\tau^+} Y_\tau f(Y_\tau) F_{D_\tau}(I_\tau + Y_\tau x_\tau)dY_\tau$ can be approximated as

$$\frac{Y_\tau^+}{6}\left[\begin{array}{c} 2\theta_{1,1} f_Y(\theta_{1,1}) F_{D_\tau}(I_\tau + \theta_{1,1}x_\tau) + 2\theta_{2,1} f_Y(\theta_{2,1}) F_{D_\tau}(I_\tau + \theta_{2,1}x_\tau) + \\ Y_\tau^+ f_Y(Y_\tau^+) F_{D_\tau}(I_\tau + Y_\tau^+ x_\tau) \end{array}\right] \tag{3.2}$$

Substitute back to equation (2.8), we obtain

$$g_1(x_\tau,\lambda) = (h_\tau + v_\tau)\frac{Y_\tau^+}{6}\left[\begin{array}{c} 2\theta_{1,1} f_Y(\theta_{1,1}) F_{D_\tau}(I_\tau + \theta_{1,1}x_\tau) + \\ 2\theta_{2,1} f_Y(\theta_{2,1}) F_{D_\tau}(I_\tau + \theta_{2,1}x_\tau) + \\ Y_\tau^+ f_Y(Y_\tau^+) F_{D_\tau}(I_\tau + Y_\tau^+ x_\tau) \end{array}\right] - v_\tau \mu_\tau + c_\tau + \lambda c_\tau = 0 \tag{3.3}$$

Then in similar fashion, we can approximate the second derivative as

$$g_1^{'}(x_\tau,\lambda) = (h_\tau + v_\tau)\frac{Y_\tau^+}{6}\left[\begin{array}{c} 2\theta_{1,1}^2 f_Y(\theta_{1,1}) f_{D_\tau}(I_\tau + \theta_{1,1}x_\tau) + \\ 2\theta_{2,1}^2 f_Y(\theta_{2,1}) f_{D_\tau}(I_\tau + \theta_{2,1}x_\tau) + \\ {Y_\tau^+}^2 f_Y(Y_\tau^+) f_{D_\tau}(I_\tau + Y_\tau^+ x_\tau) \end{array}\right] \tag{3.4}$$

3.1.2 Approximation for the second scenario

Let $(a_2 = 0, b_2 = \frac{D_\tau^+ - I_\tau}{x_\tau})$, then we calculate $\gamma_2, \theta_{1,2}, \theta_{2,2}$ as

$$\gamma_2 = \frac{(b_2 - a_2)}{3} = \frac{(D_\tau^+ - I_\tau)}{3x_\tau}$$

$$\theta_{1,2} = a_2 + \gamma_2 i = \frac{(D_\tau^+ - I_\tau)}{3x_\tau} = \gamma_2$$

$$\theta_{2,2} = \frac{2(D_\tau^+ - I_\tau)}{3x_\tau}$$

From equation 2.12 $\int_0^{\frac{D_\tau^+ - I_\tau}{x_\tau}} Y_\tau f(Y_\tau) F_{D_\tau}(I_\tau + Y_\tau x_\tau) dY_\tau$ can be approximated as

$$\frac{D_\tau^+ - I_\tau}{6x_\tau}\begin{bmatrix} 2\theta_{1,2} f_Y(\theta_{1,2}) F_{D_\tau}(I_\tau + \theta_{1,2}x_\tau) + 2\theta_{2,2} f_Y(\theta_{2,2}) F_{D_\tau}(I_\tau + \theta_{2,2}x_\tau) + \\ b_2 f_Y(b_2) F_{D_\tau}(I_\tau + b_2 x_\tau) \end{bmatrix} \qquad (3.5)$$

Let $\left(a_3 = \frac{D_\tau^+ - I_\tau}{x_\tau}, b_3 = Y_\tau^+\right)$, then we calculate $\gamma_3, \theta_{1,3}, \theta_{2,3}$ as

$$\gamma_3 = \frac{Y_\tau^+ x_\tau - (D_\tau - I_\tau)}{3x_\tau}$$

$$\theta_{1,3} = \frac{Y_\tau^+ x_\tau + 2(D_\tau^+ - I_\tau)}{3x_\tau}$$

$$\theta_{2,3} = \frac{2Y_\tau^+ x_\tau + (D_\tau^+ - I_\tau)}{3x_\tau}$$

From equation 2.12 $\int_{\frac{D_\tau^+ - I_\tau}{x_\tau}}^{Y_\tau^+} Y_\tau f(Y_\tau) dY_\tau$ can be approximated as

$$\frac{Y_\tau^+ x_\tau - (D_\tau^+ - I_\tau)}{6x_\tau}\left[2\theta_{1,3} f_Y(\theta_{1,3}) + 2\theta_{2,3} f_Y(\theta_{2,3}) + b_3 f_Y(b_3)\right] \qquad (3.6)$$

Substitute back to equation (2.12), we obtain the first derivative

$$g_2(x_\tau, \lambda) = (h_\tau + v_\tau)\left\{\begin{matrix} \frac{D_\tau^+ - I_\tau}{6x_\tau}\begin{bmatrix} 2\theta_{1,2} f_Y(\theta_{1,2}) F_{D_\tau}(I_\tau + \theta_{1,2}x_\tau) + \\ 2\theta_{2,2} f_Y(\theta_{2,2}) F_{D_\tau}(I_\tau + \theta_{2,2}x_\tau) + b_2 f_Y(b_2) F_{D_\tau}(I_\tau + b_2 x_\tau) \end{bmatrix} + \\ \frac{Y_\tau^+ x_\tau - (D_\tau^+ - I_\tau)}{6x_\tau}\left[2\theta_{1,3} f_Y(\theta_{1,3}) + 2\theta_{2,3} f_Y(\theta_{2,3}) + b_3 f_Y(b_3)\right] \end{matrix}\right\} - v_\tau \mu_\tau + c_\tau + \lambda c_\tau \qquad (3.7)$$

From equation (2.13), the second derivative approximation of expected total cost for second scenario is

$$g_2'(x_\tau, \lambda) = (h_\tau + v_\tau)\left\{\frac{D_\tau^+ - I_\tau}{6x_\tau}\begin{bmatrix} 2\theta_{1,2}^2 f_Y(\theta_{1,2}) f_{D_\tau}(I_\tau + \theta_{1,2}x_\tau) + 2\theta_{2,2}^2 f_Y(\theta_{2,2}) f_{D_\tau}(I_\tau + \theta_{2,2}x_\tau) + \\ b_2 f_Y(b_2) f_{D_\tau}(I_\tau + b_2 x_\tau) \end{bmatrix}\right\} \qquad (3.8)$$

3.2 Iterative processes for finding problem solutions based on Newton's method

The proposing algorithm can be applied for both scenarios mentioned in the previous section. Newton's Method uses a straight-line approximation to the function whose zero we wish to find. Given an initial estimate of the zero $x_{\tau,0}$; the value of the function at $x_{\tau,0}$, $g_0 = g(x_{\tau,0}, \lambda)$; and the value of the derivative at $x_{\tau,0}$, $\frac{\partial g_0}{\partial x_\tau} = g_0'(x_{\tau,0}, \lambda)$. The new approximation to the zero can be defined as:

$$x_{\tau,1} = x_{\tau,0} - \frac{g_0}{g_0'} \qquad (3.9)$$

The process continues until the change in the approximations is sufficiently small or stopping condition is satisfied. At the kth stage, we have

$$x_{\tau,k+1} = x_{\tau,k} - \frac{g_k}{g_k'} \qquad (3.10)$$

The calculation steps of the proposing algorithm are stated as follows:

Calculation steps:

1) Set $\lambda = 0$ and $k = 0$.
2) Define the value of D_τ^+ and Y_τ^+.
3) Let the value of $x_{\tau,k} = \mu_{D_\tau}$.
4) Check condition if $x_{\tau,k} \le \left(\frac{D_\tau^+ - I_\tau}{Y_\tau^+}\right)$ go to step 5 or else go to step 6.
5) Calculate the value of function $g_{1,k}(x_\tau)$ and $g_{1,k}'(x_\tau)$, using equation (3.3) and (3.4), respectively. Then go to step 7.
6) Calculate value of function $g_{2,k}(x_\tau)$ and $g_{2,k}'(x_\tau)$, using equation (3.7) and (3.8), respectively.
7) Compute new value of order quantity $(x_{\tau,k})$, using equation (3.10).
8) Check for every item if $g_{1,k}(x_\tau) = 0$ or $g_{2,k}(x_\tau) = 0$, go to step 9 or else go back to step 4 and repeat steps 4 to 7. At every iteration update value of k = k+1 and $x_{\tau,k}$.
9) Check budget constraint if it is satisfied, stop the calculation. The proper solution is obtained.

If it is not satisfied, we have to continue to the next steps.

10) Compute new Lagrange's multiplier as:

$$\lambda_{\tau,k} = \frac{\sum_{\tau=1}^{N} c_\tau x_{\tau,k} - Bg}{\sum_{\tau=1}^{N}\left(\frac{c_\tau^2}{g'(x_{\tau,k})}\right)} \qquad (3.11)$$

11) Update value of $\lambda_{\tau,k}$ then repeat steps 4 to 7.
12) Check if there are items with negative order quantity. Those items have to be eliminated then repeat steps 10 to 11.
13) If order quantities for all items are positive, stop the calculation. We obtain the final solution or else repeat step 12.

4. RESULT

This section consists of two numerical examples. The first example compares between the proposing algorithm with that of Inderfurth. As previously mentioned, he assumed that the demand and yield are uniformly distributed and budget constraint is not present. Therefore, we develop a numerical example with five items to be purchased with the similar assumptions except that the budget constraint is considered in our example. The second example is the implementation of the algorithm for a fruit retailer in a market where 5 different kinds of tropical fruits are included.

4.1 1st numerical example

The numerical data for this example are shown in the following Table 2. Demand and yield of all items are known uniform and continuous which their probability density functions can be defined as

$$f(D_\tau) = \frac{1}{D_\tau^+} \quad \text{for} \quad 0 \le D_\tau \le D_\tau^+$$

$$f(Y_\tau) = \frac{1}{Z_\tau^+} \quad \text{for} \quad 0 \le Z_\tau \le Z_\tau^+$$

The available budget for this problem is \$300.

Table 2 Numerical data

Item(τ)	$D_\tau\ (a,b)$[1]	$Y_\tau(a,b)$	h_τ	v_τ	c_τ	I_τ
1	(0, 120)	(0, 0.78)	2.5	13	2	7
2	(0, 50)	(0, 0.82)	3	10	3	2
3	(0, 45)	(0, 0.85)	1	15	3	5
4	(0, 70)	(0, 0.74)	0.5	16	6	3
5	(0, 20)	(0, 0.91)	4.5	20	10	6

[1] a is the lower bound, b is the upper bound

We implement our algorithm to solve the problem and then compare obtained solution to that of Inderfurth which shown in Table 3.

Table 3 Calculated solution comparison

Item(τ)	Order quantity		% Error
	Proposed algorithm	Inderfurth	
1	98.28	103.73	5.25%
2	14.42	15.21	5.19%
3	28.98	30.59	5.26%
4	14.50	15.30	5.23%
5	-9.09	-9.59	5.21%

From table 3, the approximated solution is close to the optimal solution with the average percentage error of 5.23%. Noted that for item#5, the order quantities obtained from both algorithms are shown as negative number for the comparison purpose which in fact they have to be round up to zero.

4.2 2nd numerical example

This example illustrates the implementation of the model to a fruit store in a market. The store sells variety kinds of fruit product but we will focus on the most five popular fruits. The numerical data of this example is shown in Table 4 and the available budget is \$150. In this problem $D_\tau^+$and $Y_\tau^+$for Normal distribution is denoted as $D_\tau^+$and $Y_\tau^+ = \mu + 3\sigma$ and for Uniform distribution $D_\tau^+$and Y_τ^+ = b.

Table 4 Numerical data

Item(τ)	D_τ		Y_τ		h_τ	v_τ	c_τ	I_τ
	Distribution type	Parameters	Distribution type	Parameters				
1	Normal (μ,σ)	(50,15)	Normal (μ,σ)	(0.75,0.2)	5	13	1	3
2	Uniform (a,b)	(0,45)	Uniform (a,b)	(0,0.85)	1	15	3	5
3	Normal (μ,σ)	(36,10)	Normal (μ,σ)	(0.6,0.15)	3	14	2	10
4	Normal (μ,σ)	(60,7)	Uniform (a,b)	(0,0.82)	0.5	11	4	2
5	Uniform (a,b)	(0,120)	Uniform (a,b)	(0,0.78)	4.5	13	2	7

After implement the algorithm, problem solution is obtained and shown in Table 5.

Table 5 Calculated Solution

Item(τ)	Orderquantity (x_τ)	Budget usage
1	59.12	59.12
2	-6.94	-20.83
3	34.80	69.60
4	1.90	7.59
5	17.26	34.52

From Table 5, one can see that the order quantity of item# 2 is a negative amount; therefore, we have to eliminate that item out of consideration and recalculate the new value of $\lambda_{2,k}$, according to step 12 in sections 3.2. The new value of $\lambda_{\tau,k}$ is 1.135. Then we obtain the order quantity for product 1, 3, 4, 5 as 58.88, 68.69, -1.27, 13.76 respectively. As one can see that, product 4 has a negative order quantity which we have to eliminate. Then, we repeat step 12 to 14, we obtain final order quantity for product 1, 3, 5 as 58.73, 34.06, and 11.58 respectively.

5. CONCLUSIONS

As literature in this arena, the stochastic single-period inventory problem with proportional costs, we could not obtain the analytical solution when demand or yield is not follow uniform distribution. Hence, we present the developed algorithm to overcome the limitation based on two main steps which are

1) the approximation of cost function by composite trapezoid numerical integration and 2) the solution procedures by the Newton's method. The algorithm is applied with a numerical example of 5 products to purchase and two types of budget constraint; binding constraint ($600) and tight constraint ($150). The obtained solution for binding constraint case shows that budget is enough to purchase all products and the total utilized budget is $600.20 which equal to budget error of 0.03 percent. For the tight constraint case, the available budget is not enough to purchase all products; hence some product must be eliminated, in this example eliminated products are product 2 and product 4, the total utilized budget is $150.01 (error of 0.01 percent). This shows that develop algorithm is accurate and easy to use with Microsoft Excel to conduct each algorithm step. Furthermore, when budget is tight the proposing method is different than others in two aspects which are 1) it does not allow negative order quantity 2) it provides the step to prioritized the order; the item ordered first is the item that give more profit, on contrary item ordered last is the least profit utility.

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7. APPENDIX

In this appendix, proofs of equation 2.8, 2.9, 2.12 and 2.13 are provided respectively.

$$\frac{\partial TC_1(x_\tau)}{\partial x_\tau} = \frac{\partial}{\partial x_\tau}\left[\sum_{\tau=1}^{N}\left\{\begin{array}{c} c_\tau x_\tau + \\ \int_0^{Y_\tau^+}\int_0^{I_\tau+Y_\tau x_\tau} h_\tau(I_\tau + Y_\tau x_\tau - D_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau + \\ \int_0^{Y_\tau^+}\int_{I_\tau+Y_\tau x_\tau}^{D_\tau^+} v_\tau(D_\tau - I_\tau - Y_\tau x_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau \end{array}\right\} + \quad \lambda\left(\sum_{\tau=1}^{N} c_\tau x_\tau - Bg\right)\right]$$

Let differentiate the equation separately for each term as follow.

$$\frac{\partial(c_\tau x_\tau)}{\partial x} = c_\tau$$

$$\frac{\partial\left(\int_0^{Y_\tau^+}\int_0^{I_\tau+Y_\tau x_\tau} h_\tau(I_\tau+Y_\tau x_\tau-D_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau\right)}{\partial x_\tau} = h_\tau \int_0^{Y_\tau^+} Y_\tau f(Y_\tau) \int_0^{I_\tau+Y_\tau x_\tau} f(D_\tau) dD_\tau dY_\tau$$

$$= h_\tau \int_0^{Y_\tau^+} Y_\tau f(Y_\tau) F(I_\tau + Y_\tau x_\tau)\, dY_\tau$$

$$\frac{\partial\left(\int_0^{Y_\tau^+}\int_0^{I_\tau+Y_\tau x_\tau} h_\tau(I_\tau+Y_\tau x_\tau-D_\tau) f(D_\tau) f(Y_\tau) dD_\tau dY_\tau\right)}{\partial x} = -v_\tau \int_0^{Y_\tau^+} Y_\tau f(Y_\tau)\left[1 - \int_0^{I_\tau+Y_\tau x_\tau} f(D_\tau) dD_\tau\right] dY_\tau$$

$$= -v_\tau \int_0^{Y_\tau^+} Y_\tau f(Y_\tau)\left[1 - F(I_\tau + Y_\tau x_\tau)\right] dY_\tau$$

$$= -v_\tau \int_0^{Y_\tau^+} Y_\tau f(Y_\tau) dY_\tau + v_\tau \int_0^{Y_\tau^+} Y_\tau f(Y_\tau) F(I_\tau + Y_\tau x_\tau) dY_\tau$$

$$= -v_\tau E[Y_\tau] + v_\tau \int_0^{Y_\tau^+} Y_\tau f(Y_\tau) F(I_\tau + Y_\tau x_\tau) dY_\tau$$

Therefore, combine all terms, we obtain equation 2.8

For equation 2.9, we applied the following derivative property, T.W. Epps [17]

$$\frac{\partial}{\partial x} F_x[h(y)] = f_x[h(y)]\frac{\partial h(y)}{\partial x}$$

then, take the second derivative we obtain equation 2.9 as follow

$$\frac{\partial^2 TC_1(x_\tau)}{\partial x_\tau^2} = g_1'(x_\tau, \lambda) = (h_\tau + v_\tau)\int_0^{Y_\tau^+} Y_\tau^2 f(Y_\tau) f_{D_\tau}(I_\tau + Y_\tau x_\tau) dY_\tau$$

Let proof equation 2.10

$$TC_2(x_\tau)=\sum_{\tau=1}^{N}\left\{\begin{array}{c} c_\tau x_\tau+\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}\int_0^{I_\tau+Y_\tau x_\tau}h_\tau(I_\tau+Y_\tau x_\tau-D_\tau)f(D_\tau)f(Y_\tau)dD_\tau dY_\tau+\\ \int_{\frac{D_\tau^+-I_\tau}{x_\tau}}^{Y_\tau^+}\int_0^{D_\tau^+}h_\tau(I_\tau+Y_\tau x_\tau-D_\tau)f(D_\tau)f(Y_\tau)dD_\tau dY_\tau+\\ \int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}\int_{I_\tau+Y_\tau x_\tau}^{D_\tau^+}v_\tau(D_\tau-I_\tau-Y_\tau x_\tau)f(D_\tau)f(Y_\tau)dD_\tau dY_\tau \end{array}\right\}+\lambda\left(\sum_{\tau=1}^{N}c_\tau x_\tau-Bg\right)$$

Let differentiate the equation separately for each term as follow.

$$\frac{\partial}{\partial x_\tau}\left[\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}\int_0^{I_\tau+Y_\tau x_\tau}h_\tau(I_\tau+Y_\tau x_\tau-D_\tau)f(D_\tau)f(Y_\tau)dD_\tau dY_\tau+\int_{\frac{D_\tau^+-I_\tau}{x_\tau}}^{Y_\tau^+}\int_0^{D_\tau^+}h_\tau(I_\tau+Y_\tau x_\tau-D_\tau)f(D_\tau)f(Y_\tau)dD_\tau dY_\tau\right]$$

$$=\ h_\tau\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau f(Y_\tau)\int_0^{I_\tau+Y_\tau x_\tau}f(D_\tau)dD_\tau dY_\tau+h_\tau\int_{\frac{D_\tau^+-I_\tau}{x_\tau}}^{Y_\tau^+}Y_\tau f(Y_\tau)\int_0^{D_\tau^+}f(D_\tau)dD_\tau dY_\tau$$

$$=\ h_\tau\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau f(Y_\tau)F(I_\tau+Y_\tau x_\tau\,)dY_\tau+h_\tau\int_{\frac{D_\tau^+-I_\tau}{x_\tau}}^{Y_\tau^+}Y_\tau f(Y_\tau)dY_\tau$$

$$\frac{\partial}{\partial x_\tau}\left[\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}\int_{I_\tau+Y_\tau x_\tau}^{D_\tau^+}v_\tau(D_\tau-I_\tau-Y_\tau x_\tau)f(D_\tau)f(Y_\tau)dD_\tau dY_\tau\right]$$

$$=\ -v_\tau\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau f(Y_\tau)\int_{I_\tau+Y_\tau x_\tau}^{D_\tau^+}f(D_\tau)dD_\tau dY_\tau$$

$$=\ -v_\tau\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau f(Y_\tau)\left[1-\int_0^{I_\tau+Y_\tau x_\tau}f(D_\tau)dD_\tau\right]dY_\tau$$

$$=\ -v_\tau\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau f(Y_\tau)dY_\tau+\ v_\tau\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau f(Y_\tau)F(I_\tau+Y_\tau x_\tau\,)dY_\tau$$

$$=\ -v_\tau\left[\int_0^{Y_\tau^+}Y_\tau f(Y_\tau)dY_\tau-\int_{\frac{D_\tau^+-I_\tau}{x_\tau}}^{Y_\tau^+}Y_\tau f(Y_\tau)dY_\tau\right]+\ v_\tau\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau f(Y_\tau)F(I_\tau+Y_\tau x_\tau\,)dY_\tau$$

$$=\ -v_\tau E[Y_\tau]+v_\tau\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau f(Y_\tau)F(I_\tau+Y_\tau x_\tau\,)dY_\tau+v_\tau\int_{\frac{D_\tau^+-I_\tau}{x_\tau}}^{Y_\tau^+}Y_\tau f(Y_\tau)dY_\tau$$

Therefore, combine all terms, we obtain equation 2.12

$$\frac{\partial TC_2(x_\tau)}{\partial x_\tau}=g_2(x_\tau,\lambda)=(h_\tau+v_\tau)\left[\begin{array}{c}\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau f(Y_\tau)F_{D_\tau}(I_\tau+Y_\tau x_\tau)dY_\tau+\\ \int_{\frac{D_\tau^+-I_\tau}{x_\tau}}^{Y_\tau^+}Y_\tau f(Y_\tau)dY_\tau\end{array}\right]-v_\tau\mu_\tau+c_\tau+\lambda c_\tau$$

then, take the second derivative we obtain equation 2.13 as follow

$$\frac{\partial^2 TC_2(x_\tau)}{\partial x_\tau^2}=g_2'(x_\tau,\lambda)=(h_\tau+v_\tau)\left[\int_0^{\frac{D_\tau^+-I_\tau}{x_\tau}}Y_\tau^2\,f(Y_\tau)f_{D_\tau}(I_\tau+Y_\tau x_\tau)dY_\tau\right]$$

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