Extensive Multilinear Algebraic Transformer Model

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ABSTRACT

An extensive s-domain multilinear algebraic model of the transformer has been proposed. This model is alternatively a tensor algebraic model because the multilinear algebra is alternatively the tensor algebra. Unlike the traditional matrix-vector approach, which relies on conventional linear algebra, this model, which in turn uses the multilinear algebra that is of higher dimension and is thus more generic, is applicable to those recently often cited transformers which often have unconventional characteristics such as frequency variant parameters, time variant parameters, and fractional impedance. Examples of such transformers are on-chip monolithic transformers, dynamic transformers, and transformers with fractional impedances. The imperfect coupling has been considered, and a multiple winding transformer has also been assumed. Applications of the proposed model to the chosen recent transformers with unconventional characteristics is presented. The effects of failure of Kron's postulate on power invariant and validity of duality invariant, which pertain to the mentioned issues, are also discussed. The proposed extensive model is more inclusive and up to date than the matrix-vector based model and previous algebraic models. However, it is more complicated.

Keywords: Dynamic Transformer, Fractional Mutual Inductance, Multilinear Algebra, On-chip Monolithic Transformer, Tensor Algebra

1. INTRODUCTION

Transformers have been used in various electrical engineering applications for decades. According to the simplicity of the algebraic based analysis in the complex frequency domain (s-domain), transformers have been traditionally modelled in the s-domain by using a classical linear algebra (matrix-vector algebra) approach where the impedances have been modelled by using linear functions. Moreover, those parameters which comprise the coefficients of the impedance functions have been assumed to be constant. Therefore this traditional approach works well with a conventional transformer which employs linear impedance functions and constant circuit parameters. Unfortunately this is not applicable to many recently developed transformers including on-chip monolithic transformers of both passive and active types [1]-[13], dynamic transformers [14]-[16] and those with fractional impedances which is termed the fractional-order mutual inductance [17]. These recently developed transformers have been applied in many electronic circuits [2]-[13], [17]. Severe error may occur in calculations using the traditional model [6]. This is because these recent transformers employ unconventional characteristics such as frequency variant circuit parameters, time variant circuit parameters, and fractional impedances, all of which are far beyond the scope of the traditional approach.

In the last few decades, a very powerful mathematical tool entitled multilinear algebra, or tensor algebra, which is the generalization of the matrix-vector algebra, has been applied to electrical engineering [18]-[24]. Tensor algebraic modelling attempts of the transformer have been proposed [19], [20], [24]. Unfortunately, the results of [19] and [20] are also inapplicable to recent transformers because these works used order 2 tensors as the impedance and current/voltage transformation arrays and used order 1 tensors for current/voltage arrays. They also used linear impedance functions and assumed that all parameters which comprise the coefficients of the current, voltage and impedance functions are neither time nor frequency dependent. Therefore these previous modelling attempts are effectively identical to the aforementioned traditional matrix-vector algebra based modelling approach. They also fail to model many modern transformers. Moreover, these previous attempts were performed assuming that the coupling is perfect, thus the coupling factor is fixed at 1. Unfortunately, the coupling can be imperfect in many recent transformers. This occurs, for example, in passive on-chip transformers because the magnetic flux linkage is weak [1]-[5], and in active on-chip transformers since lossy active couplings have been utilized [7]–[13]. Their coupling factors can be lower than 1. In practice, such coupling factors have arbitrary values between 0 and 1.

Despite the failures of [19] and [20], tensor algebra is still very promising due to its greater generality compared to matrix-vector algebra. Therefore we de-

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rived a tensor algebraic model of the on-chip monolithic transformer in [24] where the effects of imperfect coupling and frequency dependent circuit parameters, which are common to the on-chip transformer, have been taken into account. However, this previous work used order 2 tensors for the impedance and current/voltage transformation arrays and used order 1 tensors for current/voltage arrays similarly to [19] and [20]. Thus that model also fails to model some recent transformers, particularly those with time variant circuit parameters. As a result, in this paper we extend our previous work in this research by using tensors of order 2 for the current/voltage arrays, and order 3 tensors for the current/voltage transformation and the impedance arrays. Each element of these tensors refers to a single arbitrary time instant. By doing so, and using the nonlinear sdomain impedance functions, the effects of both time and frequency variant circuit parameters can be simultaneously taken into account even though the model is purely in the s-domain. The application of our extended model to the chosen recent transformers with frequency dependent parameters, time dependent parameters, and fractional impedances can be done by simply changing the model parameters according to the transformer of interest. Examples are presented in this work. Moreover, the effects of failure of Kron's postulate on power invariant and validity of duality invariant [22] to the voltage-current relationships of the transformer, which merit concern for the mentioned issues, have also been discussed. The proposed extensive tensor algebraic model is more inclusive than its predecessor [24]. It is an efficient and up to date approach for modelling recent transformers. In the subsequent sections, an overview of tensor algebra will be briefly given.

2. A BRIEF OVERVIEW OF TENSOR ALGEBRA

A tensor can be defined as a multidimensional array. Any tensor of order N is an N-dimensional array. Let $\boldsymbol{X}$ be arbitrary tensor of order N, it's Frobenius norm can be given in terms of its element $\boldsymbol{x}_{i_1,i_2,\ldots,i_N}$ as

$$\|\boldsymbol{X}\|_F = \sqrt{\sum_{i_N=1}^{I_N}\cdots\sum_{i_2=1}^{I_2}\sum_{i_1=1}^{I_1}[\mathbf{x}^2_{i_1,i_2,\ldots i_N}]} \quad (1)$$

where $\{i_1\} = \{1, 2, \ldots, I_1\}$, $\{i_2\} = \{1, 2, \ldots, I2\}, \ldots$, $\{i_N\} = \{1, 2, \ldots, I_N\}$.

If $\boldsymbol{Y}$ is another tensor with order N as well as $\boldsymbol{X}$, their scalar product i.e. $< \boldsymbol{X}, \boldsymbol{Y} >$, is

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \sum_{i_N=1}^{I_N}\cdots\sum_{i_2=1}^{I_2}\sum_{i_1=1}^{I_1}[\boldsymbol{x}_{i_1,i_2,\ldots i_N}\boldsymbol{y}_{i_1,i_2,\ldots i_N}] \quad (2)$$

where $\boldsymbol{y}_{i_1,i_2,\ldots i_N}$ denotes an arbitrary element of $\boldsymbol{Y}$.

Moreover, if $\boldsymbol{X}$ and $\boldsymbol{Y}$ also have the same dimensions, their Hadamard product [26] $\boldsymbol{Z} = \boldsymbol{Y} \otimes \boldsymbol{Y}$ is an order N tensor with similar dimensions to those of $\boldsymbol{X}$ and $\boldsymbol{Y}$ where each of its elements can be given in terms of $\boldsymbol{x}_{i_1,i_2,\ldots i_N}$ and $\boldsymbol{y}_{i_1,i_2,\ldots i_N}$ by

$$\boldsymbol{z}_{i_1,i_2,\ldots,i_N,j_1,j_2,\ldots j_N} = \boldsymbol{x}_{i_1,i_2\ldots i_N}\boldsymbol{y}_{i_1,i_2,\ldots i_N} \quad (3)$$

Now let $\boldsymbol{Y}$ be of arbitrary order M with $\boldsymbol{y}_{i_1,i_2,\ldots i_M}$ with $\{j_1\} = \{1, 2, \ldots, J_1\}$, $\{j_2\} = \{1, 2, \ldots, J_2\}, \ldots$, $\{j_M\} = \{1, 2, \ldots, J_M\}$ as its arbitrary element. The mode p contraction product of $\boldsymbol{X}$ and $\boldsymbol{Y}$, $\boldsymbol{Z} = \boldsymbol{X} \times_p \boldsymbol{Y}$, is an order $M + N - 2$ tensor where each of its elements can be given in terms of $\boldsymbol{x}_{i_1,i_2,\ldots i_N}$ and $\boldsymbol{y}_{i_1,i_2,\ldots i_M}$ by

$$\boldsymbol{z}_{i_1,i_2,\ldots i_{p-1},i_{p+1},\ldots,i_N,j_1,j_2,\ldots j_{p-1},j_{p+1},\ldots j_M} = \sum_{i_p=1}^{I_p}[\boldsymbol{x}_{i_1,i_2,\ldots i_N}\boldsymbol{y}_{j_1,j_2,\ldots j_M}] \quad (4)$$

For a detailed foundation of tensor algebra, readers may refer to tutorials on tensor algebra [25]-[27].

3. THE PROPOSED MODEL

In this section, the proposed extensive tensor algebraic model will be derived. For the conventional transformer, the current/voltage arrays can be defined as vectors of s-domain current/voltage functions. Moreover, the current/voltage transformation and impedance arrays can be defined as matrices of time invariant transformer circuit parameters and linear constant coefficient impedance functions respectively, as these coefficients are comprised of such time invariant parameters. However, this is not the case for some recent transformers. If the current/voltage arrays of such a transformer have been defined as vectors where the current/voltage transformation and impedance arrays have been defined as matrices, the time dependent parameters yield current/voltage transformation matrices with time dependent elements and the impedance functions with time dependent coefficients. As a result, the elements of current/voltage vectors become functions of both complex frequency and time. Thus, the obtained model will be in both the time domain and the s-domain. Obviously, performing any task using such a model can be very complicated.

Therefore we redefine the current/voltage arrays of both primary and secondary windings as order 2 tensors. For an arbitrary transformer with N primary and M secondary windings, the current tensors of the primary and secondary windings are respectively $\boldsymbol{I_{Pik}}$ and $\boldsymbol{I_{Sjk}}$ where the voltage tensors of both primary and secondary windings are $\boldsymbol{V_{Pik}}$ and $\boldsymbol{V_{Sjk}}$. Note that $\{i\} = \{1, 2, \ldots, N\}$ and

$\{j\} = \{1, 2, \ldots, M\}$ where N and M are integers. Many time instances must be considered due to the effects of time dependent transformer parameters: $\{k\} = \{1, 2, \ldots, K\}$ where K can be arbitrary integer. Thus each, element of $\boldsymbol{I_{Pik}}$, I_{Pik}, and that of $\boldsymbol{I_{Sjk}}$, I_{Sik}, are respectively the s-domain current function at arbitrary k$^{\text{th}}$ instant of any i$^{\text{th}}$ primary winding and that of any j$^{\text{th}}$ secondary winding. For $\boldsymbol{V_{Pik}}$ and $\boldsymbol{V_{Sjk}}$, we let each of their elements, V_{Pik} and V_{Sjk}, be respectively the s-domain voltage function of any i$^{\text{th}}$ primary winding and that of any j$^{\text{th}}$ secondary winding at any k$^{\text{th}}$ instant. As a result, I_{Pik}, I_{Sik}, V_{Pik} and V_{Sjk} are functions of complex frequency only, and the obtained current/voltage tensors are able to model the time dependencies of current/voltage functions caused by those of the circuit parameters despite the fact that these tensors are purely in the s-domain. This cannot be achieved if an order 1 tensor has been assumed similarly to [19], [20] and [24] because a tensor of order 1 is mathematically equivalent a vector. Before we proceed further, it should be mentioned here that the dimensions of $\boldsymbol{I_{Pik}}$ and $\boldsymbol{V_{Pik}}$ are $N\times K$, but those of $\boldsymbol{I_{Sjk}}$ and $\boldsymbol{V_{Sjk}}$ are $M \times K$.

If we let $\boldsymbol{I_{Pi}}$ and $\boldsymbol{I_{Sj}}$ be extracted from arbitrary k$^{\text{th}}$ mode-1 fiber [25] of $\boldsymbol{I_{Pik}}$ and $\boldsymbol{I_{Sjk}}$, the current transformations by from i$^{\text{th}}$ primary to j$^{\text{th}}$ secondary winding and vice versa can be modelled in terms of the contraction products [26] as

$$\forall k[\boldsymbol{I_{Sj}} = \boldsymbol{I_{Pi}} \times_1 \boldsymbol{C_{ji}^{SP}} \tag{5}$$

$$\forall k[\boldsymbol{I_{Pi}} = \boldsymbol{I_{Si}} \times_1 \boldsymbol{C_{ij}^{PS}} \tag{6}$$

where $\times_1$ denote the mode 1 contraction product operator. Moreover, $\boldsymbol{C_{ji}^{SP}}$ and $\boldsymbol{C_{ij}^{PS}}$ have been extracted from the arbitrary k$^{\text{th}}$ frontal frontal slice [25] of $\boldsymbol{C_{ji}^{SP}}$ and $\boldsymbol{C_{ij}^{PS}}$ which are the primary to secondary current transformation tensors and vice versa, respectively.

For taking the effects of time variant parameters into account while keeping the model purely in the s-domain for simplicity, $\boldsymbol{C_{ji}^{SP}}$ and $\boldsymbol{C_{ij}^{PS}}$ must be of order 3. Their dimensions are $M \times N \times K$ and $N \times M \times K$. Each of their elements, c_{ij}^{PS} and c_{ji}^{SP} , which are simply time invariant, can be respectively given by

$$c_{jik}^{SP} = -\frac{k_{jik}^{SP}}{n_{jik}^{SP}} \tag{7}$$

$$c_{ijk}^{PS} = -\frac{k_{ijk}^{PS}}{n_{ijk}^{PS}} \tag{8}$$

where n_{ijk}^{PS} (n_{jik}^{SP}) and k_{ijk}^{PS} (k_{jik}^{SP}) denote the turn ratios and coupling factors from j$^{\text{th}}$ secondary winding to i$^{\text{th}}$ primary winding (vice versa) at an arbitrary k$^{\text{th}}$ instant. The effects of imperfect coupling have been taken into account, so $0 < k_{ijk}^{PS} \leq 1$ and $0 < k_{jik}^{SP} \leq 1$.

If we let these current transformation tensors be of order 2 similarly to those of [19], [20] and [24], they will be mathematically equivalent to the matrices. As a result, their elements must be time dependent instead of being constant for modelling the the effects of time variant parameters. As a result, the obtained model will be in both the time domain and the s-domain, which is complicated.

If we let $\boldsymbol{V_{Pi}}$ and $\boldsymbol{V_{Sj}}$ be extracted from an arbitrary k$^{\text{th}}$ mode-1 fiber of $\boldsymbol{V_{ik}}$ and $\boldsymbol{V_{jk}}$, the voltage transformation from i$^{\text{th}}$ primary to j$^{\text{th}}$ secondary winding and vice versa can be modelled as follows

$$\forall k[\boldsymbol{V_{Sj}} = \boldsymbol{V_{Pi}} \times_1 \boldsymbol{A_{ji}^{SP}}] \tag{9}$$

$$\forall k[\boldsymbol{V_{Pi}} = \boldsymbol{V_{Sj}} \times_1 \boldsymbol{A_{ij}^{PS}}] \tag{10}$$

where $\boldsymbol{A_{ji}^{SP}}$ and $\boldsymbol{A_{ij}^{PS}}$ have been extracted from the arbitrary k$^{\text{th}}$ frontal frontal slices of $\boldsymbol{A_{jik}^{SP}}$ and $\boldsymbol{A_{ijk}^{PS}}$ which are the primary to secondary voltage transformation tensor and vice versa, respectively.

Since $\boldsymbol{C_{jik}^{SP}}$ and $\boldsymbol{C_{ijk}^{PS}}$ are of order 3, so are $\boldsymbol{A_{jik}^{SP}}$ and $\boldsymbol{A_{ijk}^{PS}}$. Moreover, $\boldsymbol{A_{jik}^{SP}}$ and $\boldsymbol{A_{ijk}^{PS}}$ can be respectively related to $\boldsymbol{C_{jik}^{SP}}$ and $\boldsymbol{C_{ijk}^{PS}}$ by the following

$$\forall k[(\boldsymbol{A_{ij}^{PS}})^T \times_1 \boldsymbol{C_{ji}^{SP}} = \boldsymbol{I_{N\times N}}] \tag{11}$$

$$\forall k[(\boldsymbol{A_{ji}^{SP}})^T \times_1 \boldsymbol{C_{ij}^{PS}} = \boldsymbol{I_{M\times M}}] \tag{12}$$

where $(\ )^T$ denotes the transpose operator. Moreover, $\boldsymbol{I_{M\times M}}$ and $\boldsymbol{I_{N\times N}}$ are identity order 2 tensors with $M\times M$ and $N\times N$ as their dimensions respectively. Note also that the the dimensions of $\boldsymbol{A_{ji}^{SP}}$ and $\boldsymbol{A_{ij}^{PS}}$ are $M\times N\times K$ and $N\times M\times K$.

Similarly to the current/voltage arrays, the impedance arrays of both self and mutual impedances must be redefined for similar reasons. Therefore they become tensors of order 3. For covering all impedances of the transformer, a total of 4 impedance tensors must be formulated. These impedance tensors are the self impedance tensor of primary windings, that of secondary winding the mutual impedance tensors of coupling from secondary to primary winding, and that of coupling in the inverse direction. We respectively denote them by $\boldsymbol{Z_{i_1i_2k}^{sP}}$, $\boldsymbol{Z_{j_1j_2k}^{sS}}$, $\boldsymbol{Z_{ijk}^{mPS}}$ and $\boldsymbol{Z_{jik}^{mSP}}$, where $\{i_1\} = \{1, 2, \ldots, N\}$, $\{i_2\} = \{1, 2, \ldots, N\}$, $\{j_{1]\}=\{1,2,\ldots,M\}}$ and $\{j2\} = \{1, 2, \ldots, M\}$. Their dimensions are respectively $N\times N\times K$, $M\times M\times K$, $N\times M\times K$ and $M\times N\times K$. Their arbitrary elements, which can be denoted respectively by Z_{ijk}^{sP}, $Z_{j_1j_2k}^{sS}$, Z_{ijk}^{mPS}, and Z_{jik}^{mSP}, are the transformer's impedance functions at an arbitrary k$^{\text{th}}$ instant. Therefore these functions are functions of complex frequency only and have time independent coefficients. As a result, time dependencies of the transformer's parameters have been taken into account while maintaining the impedance tensors purely in the s-domain and so is the obtained model. This cannot be achieved if order 2 impedance tensors are used similarly to [19], [20] and [24] because a tensor of order 2 is mathematically equivalent to a matrix.

For including the effects of frequency dependent transformer parameters such as frequency dependent resistances and inductances [7], and fractional impedances, nonlinear impedance functions must be adopted. In order to do so, we let Z^{sP}_{ijk}, $Z^{sS}_{j_1j_2k}$, Z^{mPS}_{ijk}, and Z^{mSP}_{jik} be defined as the s-domain arbitrary order rational polynomial functions as given by (13)-(16) where $\{q_{sPk}\} = \{0, 1, 2, \ldots, Q_{sPk}\}$, $\{q_{sSk}\} = \{0, 1, 2, \ldots, Q_{sSk}\}$, $\{q_{mPSk}\} = \{0, 1, 2, \ldots, Q_{mPSk}\}$, $q_{mSPk}\} = \{0, 1, 2, \ldots, Q_{mSPk}\}$, $\{r_{sPk}\} = \{0, 1, 2, \ldots, R_{sPk}\}$, $\{r_{sSk}\} = \{0, 1, 2, \ldots, R_{sSk}\}$, $\{r_{mPSk}\} = \{0, 1, 2, \ldots, R_{mPSk}\}$ and $\{r_{mSPk}\} = \{0, 1, 2, \ldots, R_{mSPk}\}$. Note also that Q_{sPk},Q_{sSk}, Q_{mPSk}, $QmSPk$, R_{sPk}, R_{sSk}, R_{mPSk} and R_{mSPk} can be arbitrary integers. Moreover, $\alpha_{q_{sPk}}$, $\alpha_{q_{sSk}}$, $\alpha_{q_{mPSk}}$, $\alpha_{q_{mSPk}}$, $\beta_{r_{sPk}}$, $\beta_{r_{sSk}}$, $\beta_{r_{mPSk}}$ and $\beta_{r_{mSPk}}$ are neither time nor frequency dependent. The s-domain rational polynomial function has been chosen as it can model any nonlinear function. This includes the fractional order polynomial function of the fractional impedance. However, the integer order approximation methodologies of fractional Laplacian operator such as Oustaloup's approximation, regular Newton process, and continued fraction expansion [28] must be used for modelling such a fractional order polynomial function.

$$Z^{sP}_{i_1i_2k} = \begin{cases} \sum\limits_{q_{sPk}=0}^{Q_{sPk}} [\alpha_{qsPk} s^{q_{sPk}}] / \sum\limits_{r_{sPk}=0}^{R_{sPk}} [\beta_{r_{sPk}} s^{r_{sPk}}] & ; i_1 = i_2 \\ 0 & ; i_1 \neq i_2 \end{cases} \tag{13}$$

$$Z^{sS}_{j_1j_2k} = \begin{cases} \sum\limits_{q_{sSk}=0}^{Q_{sSk}} [\alpha_{qsSk} s^{q_{sSk}}] / \sum\limits_{r_{sSk}=0}^{R_{sSk}} [\beta_{r_{sSk}} s^{r_{sSk}}] & ; j_1 = j_2 \\ 0 & ; j_1 \neq j_2 \end{cases} \tag{14}$$

$$Z^{mPS}_{ijk} = \sum_{q_{mPSkl0}}^{Q_{mPSk}} [\alpha_{q_{mPSk}} s^{q_{mPSk}}] / \sum_{r_{mPSk}=0}^{R_{mPSk}} [\beta_{r_{mPSk}} s^{r_{mPSk}}] \tag{15}$$

$$Z^{mSP}_{jik} = \sum_{q_{mSPk}=0}^{Q_{mSPk}} [\alpha_{q_{mSPk}} s^{q_{mSPk}}] / \sum_{r_{mSPk}=0}^{R_{mSPk}} [\beta_{r_{mSPk}} s^{r_{mSPk}}] \tag{16}$$

If we let $\boldsymbol{Z^{sP}_{i_1i_2}}$, $\boldsymbol{Z^{sS}_{j_1j_2}}$, $\boldsymbol{Z^{mPS}_{ij}}$ and $\boldsymbol{Z^{mSP}_{ji}}$ be extracted from the arbitrary k[th] frontal frontal slices of $\boldsymbol{Z^{sP}_{i_1i_2k}}$, $\boldsymbol{Z^{sS}_{j_1j_2k}}$, $\boldsymbol{Z^{mPS}_{ijk}}$ and $\boldsymbol{Z^{mSP}_{jik}}$ respectively, the tensor algebraic voltage-current relationships of the transformer can be given by

$$\forall k[\boldsymbol{V_{Pi}} = (\boldsymbol{I_{Pi}} \times_1 \boldsymbol{Z^{sP}_{i_1i_2}}) + (\boldsymbol{I_{Sj}} \times_1 \boldsymbol{Z^{mPS}_{ij}})] \tag{17}$$

$$\forall k[\boldsymbol{V_{Sj}} = (\boldsymbol{I_{Sj}} \times_1 \boldsymbol{Z^{sS}_{j_1j_2}}) + (\boldsymbol{I_{Pi}} \times_1 \boldsymbol{Z^{mSP}_{ji}})] \tag{18}$$

which can be alternatively expressed for modelling the voltage-current characteristic of each terminal as

$$\forall k[\boldsymbol{V_{Pi}} = \boldsymbol{I_{pi}} \times_1 \boldsymbol{Z^{eP}_{i_1i_2}}] \tag{19}$$

$$\forall k[\boldsymbol{V_{Sj}} = \boldsymbol{I_{Sj}} \times_1 \boldsymbol{Z^{eS}_{j_1j_2}}] \tag{20}$$

where $\boldsymbol{Z^{eP}_{i_1i_2}}$ and $\boldsymbol{Z^{eS}_{j_1j_2}}$ have been respectively extracted from the arbitrary k[th] frontal frontal slices of $\boldsymbol{Z^{eP}_{i_1i_2k}}$ and $\boldsymbol{Z^{eS}_{j_1j_2k}}$ which are the effective impedance tensors of primary and secondary windings. Their dimensions are $N \times N \times K$ and $M \times M \times K$ respectively.

Since the effective impedance at any winding is the summation of its self impedance and the transformed mutual impedance, we can define $\boldsymbol{Z^{eP}_{i_1i_2k}}$ and $\boldsymbol{Z^{eS}_{j_1j_2k}}$ by the following tensor algebraic equations:

$$\forall k[\boldsymbol{Z^{eP}_{i_1i_2}} = \boldsymbol{Z^{sP}_{i_1i_2}} + (\boldsymbol{Z^{mPS}_{ij}} \times_1 \boldsymbol{C_{i_1i_2}} \times_2 \boldsymbol{C^{PS}_{ij}})] \tag{21}$$

$$\forall k[\boldsymbol{Z^{eS}_{j_1j_2}} = \boldsymbol{Z^{sS}_{j_1j_2}} + (\boldsymbol{Z^{mSP}_{ji}} \times_1 \boldsymbol{C_{j_1j_2}} \times_2 \boldsymbol{C^{SP}_{ji}})] \tag{22}$$

where $\times_2$ stand for the mode 2 contraction product operator. Moreover, $\boldsymbol{C_{i_1i_2}}$ and $\boldsymbol{C_{j_1j_2}}$ have been respectively extracted from the arbitrary k[th] frontal slices of $\boldsymbol{C_{i_1i_2k}}$ and $\boldsymbol{C_{j_1j_2k}}$ respectively. The dimensions of $\boldsymbol{C_{i_1i_2k}}$ and $\boldsymbol{C_{j_1j_2k}}$ are respectively $N \times N \times K$ and $M \times M \times K$ whereas their elements can be either 1 or 0. In particular, each of the elements of $\boldsymbol{C_{i_1i_2k}}$ ($\boldsymbol{C_{j_1j_2k}}$) if and only if $i_1 = i_2 (j_1 = j_2)$ and vice versa. It should be mentioned here that $\boldsymbol{C_{i_1i_2k}}$ and $\boldsymbol{C_{j_1j_2k}}$ must be introduced because two transformation tensors are necessary for transforming a mutual the impedance tensor.

At this point, the proposed tensor algebraic model of a transformer has been finally derived as a set of tensor algebraic equations where (5) and (6) model the current transformations. Moreover, (9) and (10) ((21) and (22)) model the voltage (impedance) transformations, and (19) and (20) model the corresponding terminal voltage-current characteristics. Finally, (7), (8) and (13)-(16) define the corresponding tensor elements. The proposed model can still be applied to the conventional transformer by letting $K = 1$, $Q_{sPk} = 1$, $Q_{sSk} = 1$, $Q_{mPSk} = 1$, $Q_{mSPk} = 1$, $R_{sPk} = 0$, $R_{sSk} = 0$, $R_{mPSk} = 0$ and $R_{mSPk} = 0$.

In the subsequent section, the proposed model will be applied for determining $\boldsymbol{I_{Pik}}$'s, $\boldsymbol{I_{Sjk}}$'s and $\boldsymbol{V_{Sjk}}$'s of some chosen recent transformers assuming that

$\boldsymbol{V_{Pik}}$'s have been given.

4. EXAMPLES OF APPLICATION

By applying our extensive proposed model and keeping in mind that $i_1 = i_2 = i$ and $j_1 = j_2 = j$, I_{Pik}'s, I_{Sjk}'s and V_{Sjk}'s of an arbitrary transformer with N primary and M secondary windings can be given in terms of known V_{Pik}'s as

$$I_{Pik} = (Z^{sP}_{ii} - \sum_{j=1}^{M}\sum_{i=1}^{N} \frac{k^{PS}_{ijk}}{n^{PS}_{ijk}} Z^{mPS}_{ij})^{-1} V_{Pik} \qquad (23)$$

$$I_{Sjk} = \frac{k^{SP}_{jik} V_{Pik}}{n^{SP}_{jik}(Z^{sP}_{ii} - \sum_{j=1}^{M}\sum_{i=1}^{N} \frac{k^{PS}_{ijk}}{n^{PS}_{ijk}} Z^{mPS}_{ij})^{-1} V_{Pik}} \qquad (24)$$

$$V_{Sjk} = \frac{k^{SP}_{jik} V_{Pik}}{n^{SP}_{jik}} \frac{Z^{sP}_{ii} - \sum_{j=1}^{M}\sum_{i=1}^{N} \frac{k^{SP}_{jik}}{n^{SP}_{jik}} Z^{mSP}_{ji}}{Z^{sP}_{ii} - \sum_{j=1}^{M}\sum_{i=1}^{N} \frac{k^{PS}_{ijk}}{n^{PS}_{ijk}} Z^{mPS}_{ij}} \qquad (25)$$

Here, 3 recent transformers, CMOS gyrator-C based on-chip monolithic active transformer [7], dynamic transformer for flux-trapping flux-compression generator (FT-FCG) [16], and fractional-order mutual inductance, have been chosen. These transformers respectively have frequency dependent parameters, time dependent parameters, and fractional impedances. First, the CMOS gyrator-C based on-chip transformer which has been treated in [24] will be reconsidered using our extensive tensor algebraic model which is more inclusive. This CMOS transformer employs a single input and double outputs. Its frequency dependent parameters are the resistances and inductances of the windings which are linear functions of frequency. The schematic diagram and circuit realization of this transformer is depicted in Fig. 1, which clearly shows that N =1 and M = 2.

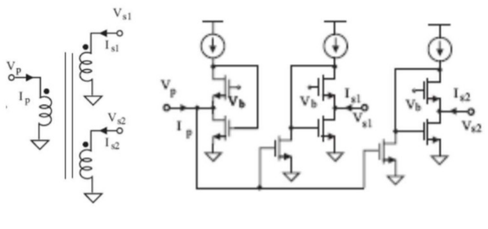

Fig.1: *Schematic diagram of the CMOS gyrator-C based single input/double outputs on-chip monolithic active transformer (left) and its circuit realization (right) [7] .*

Since this transformer has unidirectional couplings from primary to secondary windings only, we have found that

$$Z^{mPS}_{11k} = 0 \qquad (26)$$

$$Z^{mPS}_{12k} = 0 \qquad (27)$$

Therefore we have

$$I_{P1k} = \frac{V_{P1k}}{Z^{sP}_{11k}} \qquad (28)$$

$$I_{S1k} = \frac{k^{SP}_{11k} V_{P1k}}{n^{SP}_{11k} Z^{sP}_{11k}} \qquad (29)$$

$$I_{S2k} = \frac{k^{SP}_{21k} V_{P1k}}{n^{SP}_{21k} Z^{sP}_{11k}} \qquad (30)$$

However, couplings from primary to secondary windings do exist. Thus we have

$$V_{S1k} = \frac{V_{P1k}}{Z^{sP}_{11k}} \left(\frac{k^{SP}_{11k} Z^{sS}_{11k}}{n^{SP}_{11k}} - Z^{mSP}_{11k} \right) \qquad (31)$$

$$V_{S2k} = \frac{V_{P1k}}{Z^{sP}_{11k}} \left(\frac{k^{SP}_{21k} Z^{sS}_{21k}}{n^{SP}_{21k}} - Z^{mSP}_{21k} \right) \qquad (32)$$

Since the resistances and inductances this transformer are linear functions of frequency, they can be given in the s-domain by $R^{sP}_{11k} = \hat{R}^{sP}_{11k}s$, $R^{sS}_{11k} = \hat{R}^{sS}_{11k}s$, $R^{sS}_{22k} = \hat{R}^{sS}_{22k}s$, $R^{mSP}_{11k} = \hat{R}^{mSP}_{11k}s$, $R^{mSP}_{21k} = \hat{R}^{mSP}_{21k}s$, $L^{sP}_{11k} = \hat{L}^{sP}_{11k}s$, $L^{sS}_{11k} = \hat{L}^{sS}_{11k}s$, $L^{sS}_{22k} = \hat{L}^{sS}_{22k}s$, $L^{mSP}_{11k} = \hat{L}^{mSP}_{11k}s$ and $R^{mSP}_{21k} = \hat{R}^{mSP}_{21k}s$ where $\hat{R}^{sP}_{11k}$, $\hat{R}^{sS}_{11k}$, $\hat{R}^{sS}_{22k}$, $\hat{R}^{mSP}_{11k}$, $R^{mSP}_{21k} = \hat{R}^{mSP}_{21k}s$, $\hat{L}^{sP}_{11k}$, $\hat{L}^{sS}_{11k}$, $\hat{L}^{sS}_{22k}$, $\hat{L}^{mSP}_{11k}$ and $\hat{R}^{mSP}_{21k}$ are neither time nor frequency dependent. As a result, Z^{sP}_{11k} , Z^{sS}_{11k} , Z^{sS}_{22k} , Z^{mSP}_{11k} and Z^{mSP}_{21k} become nonlinear functions and can be given as follows

$$Z^{sP}_{11k} = \hat{L}^{sP}_{11k}s^2 + (L^{sP}_{11k} + \hat{R}^{sP}_{11k})s + R^{sP}_{11k} \qquad (33)$$

$$Z^{sP}_{11k} = \hat{L}^{sP}_{11k}s^2 + (L^{sS}_{11k} + \hat{R}^{sS}_{11k})s + R^{sS}_{11k} \qquad (34)$$

$$Z^{sP}_{22k} = \hat{L}^{sP}_{22k}s^2 + (L^{sS}_{22k} + \hat{R}^{sS}_{22k})s + R^{sS}_{22k} \qquad (35)$$

$$Z^{mSP}_{11k} = \hat{L}^{mSP}_{11k}s^2 + (L^{mSP}_{11k} + \hat{R}^{mSP}_{11k})s + R^{mSP}_{11k} \qquad (36)$$

$$Z^{mSP}_{21k} = \hat{L}^{mSP}_{21k}s^2 + (L^{mSP}_{21k} + \hat{R}^{mSP}_{21k})s + R^{mSP}_{21k} \qquad (37)$$

which are in terms of (13)-(16) with $Q_{sPk} = 2$, $Q_{sSk} = 2$, $Q_{mPSk} = 2$, $Q_{mSPk} = 2$, $R_{sPk} = 0$, R_{sSk}

$= 0$, $R_{mPSk} = 0$ and $R_{mSPk} = 0$.

Moreover, all coefficient terms of (33)-(37) are neither time nor frequency dependent as they are in terms of $\hat{R}^{sP}_{11k}$, $\hat{R}^{sS}_{11k}$, $\hat{R}^{sS}_{22k}$, $\hat{R}^{mSP}_{11k}$, $R^{mSP}_{21k} = \hat{R}^{mSP}_{21k}s$, $\hat{L}^{sP}_{11k}$, $\hat{L}^{sS}_{11k}$, $\hat{L}^{sS}_{22k}$, $\hat{L}^{mSP}_{11k}$ and $\hat{R}^{mSP}_{21k}$ which, in turn, depend on the MOSFET's biasing condition dependent parameters. Those include transconductance, g_m, which is a function of bias current, the parasitic capacitances such as the gate to source capacitance, C_{gs} and drain to source conductance, g_{ds} and cut-off frequency, f_T etc. [7], [28]. This is because MOSFET is the basis component of this candidate on-chip transformer. Note that k^{sP}_{11k} , k^{sP}_{21k} and n^{sP}_{21k} depend on the MOSFET's biasing condition dependent parameters [7] for a similar reason. By these dependencies, the values of $\hat{R}^{sS}_{11k}$, $\hat{R}^{sS}_{22k}$, $\hat{R}^{mSP}_{11k}$, $R^{mSP}_{21k} = \hat{R}^{mSP}_{21k}s$, $\hat{L}^{sP}_{11k}$, $\hat{L}^{sS}_{11k}$, $\hat{L}^{sS}_{22k}$, $\hat{L}^{mSP}_{11k}$, $\hat{R}^{mSP}_{21k}$,k^{sP}_{11k} , k^{sP}_{21k} and n^{sP}_{21k} can be changed with respect to the biasing condition of MOSFET. If a CMOS transformer with different topology is to be considered, other related parameters of our model can be adjusted in response, such as changing $N = 2$ which must be applied instead of $N = 1$ if the transformer has 2 primary windings.

Since the parameters of this on-chip transformer are time independent, so are its s-domain current functions, voltage functions, and impedance functions. Therefore $I_{p1} = I_{p2} \ldots = I_{pK}$, $I_{s11} = I_{s12} \ldots = I_{s1K}$, $I_{s21} = I_{s22} \ldots = I_{s2K}$, $V_{s11} = V_{s12} \ldots = V_{s1K}$, $V_{s21} = V_{s22} \ldots = V_{s2K}$, $Z^{sP}_{111} = Z^{sP}_{112} \ldots = Z^{sP}_{11K}$, $Z^{sS}_{111} = Z^{sS}_{112} \ldots = Z^{sS}_{11K}$, $Z^{sS}_{221} = Z^{sS}_{222} \ldots = Z^{sS}_{22K}$, $Z^{mSP}_{111} = Z^{mSP}_{112} \ldots = Z^{mSP}_{11K}$ and $Z^{mSP}_{211} = Z^{mSP}_{212} \ldots = Z^{mSP}_{21K}$. By using (28)-(32) with all impedance functions as given by (33)-(37), our extensive model based frequency responses of I_{P1k}, I_{S1k}, I_{S2k}, V_{S1k} and V_{S2k} can be determined and compared to their BSIM3v3 based benchmarks obtained from SPICE simulation of the transformer as depicted in Fig. 2-6. Here, the 0.35μm level CMOS process technology of AMS has been chosen as the model parameterization and simulation basis since this transformer has been realized in the above 100 nm regime. Similarly to [24], we use the aspect ratios of 111.11 and 55.56 for the winding transistors and coupling transistors respectively. These transistors have been biased by $V_b = 1.4$ V. Moreover, a sinusoidal waveform with the magnitude of 20mV has been chosen as the input voltage and the simulation frequency has been ranged from 1Hz to 100 MHz which lies within the operating range of the transformer [7]. From Fig. 2-6, strong agreements between the model based responses and their corresponding BSIM3v3 based benchmarks can be observed for several decades of frequency. At this point, it can be seen that our extensive model is very applicable to the CMOS gyratorC based on-chip transformer similarly to its predecessor [24] where more detailed analysis has been proposed in this work.

Consider the candidate dynamic transformer which has a single input and single output [16]. It can be seen that $N = 1$ and $M = 1$ for this transformer. The schematic diagram of FT-FCG can be

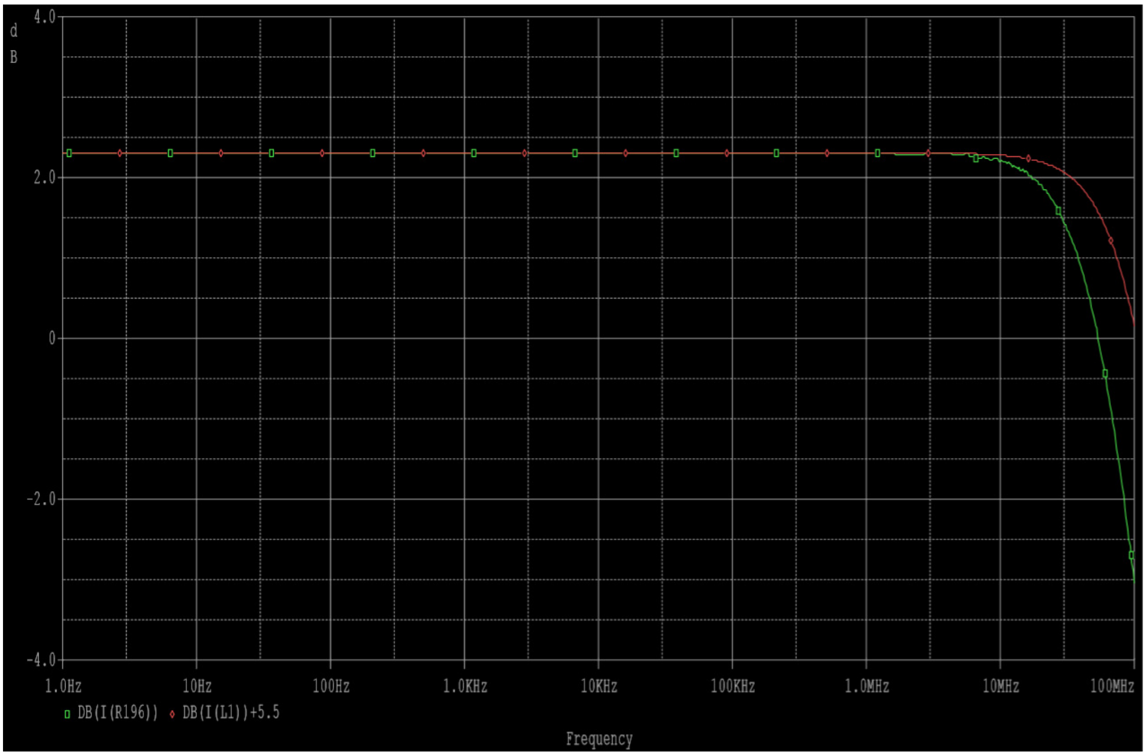

Fig.2: *Comparative frequency responses of I_{P1k}: Model based response ($\Diamond$), BSIM3v3 based benchmark ($\Box$).*

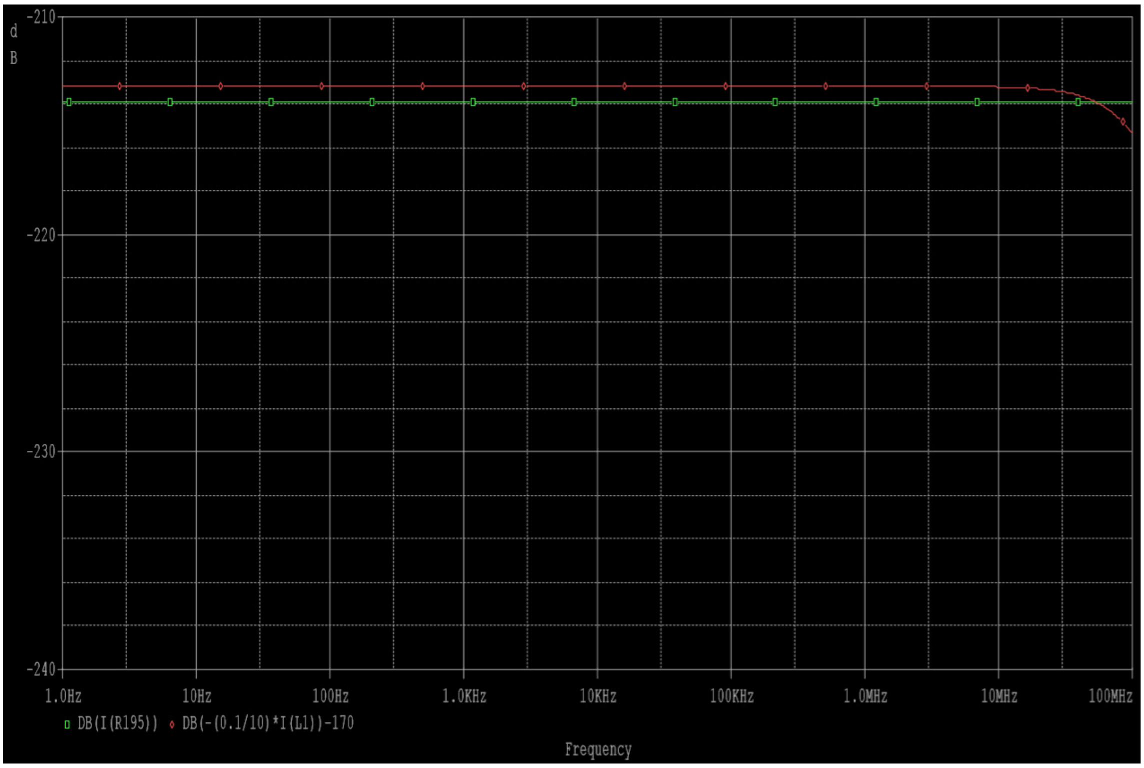

Fig.3: *Comparative frequency responses of I_{S1k}: Model based response (◇), BSIM3v3 based benchmark (□).*

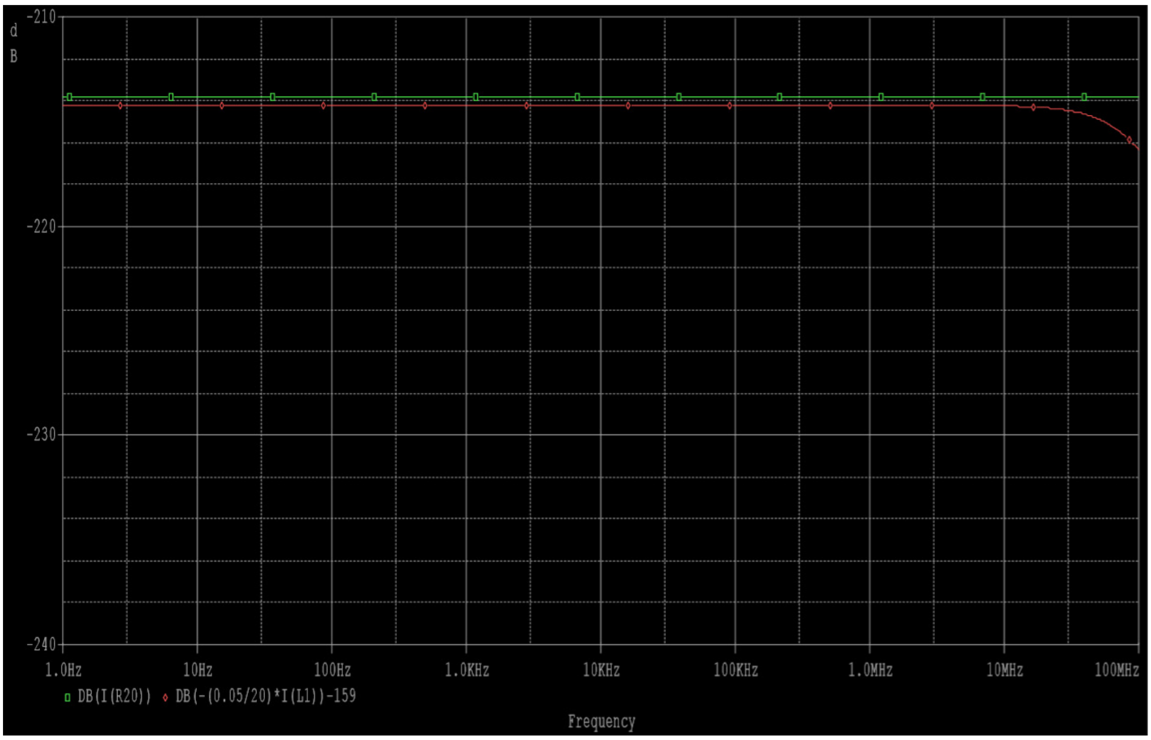

Fig.4: *Comparative frequency responses of I_{S2k}: Model based response (◇), BSIM3v3 based benchmark (□).*

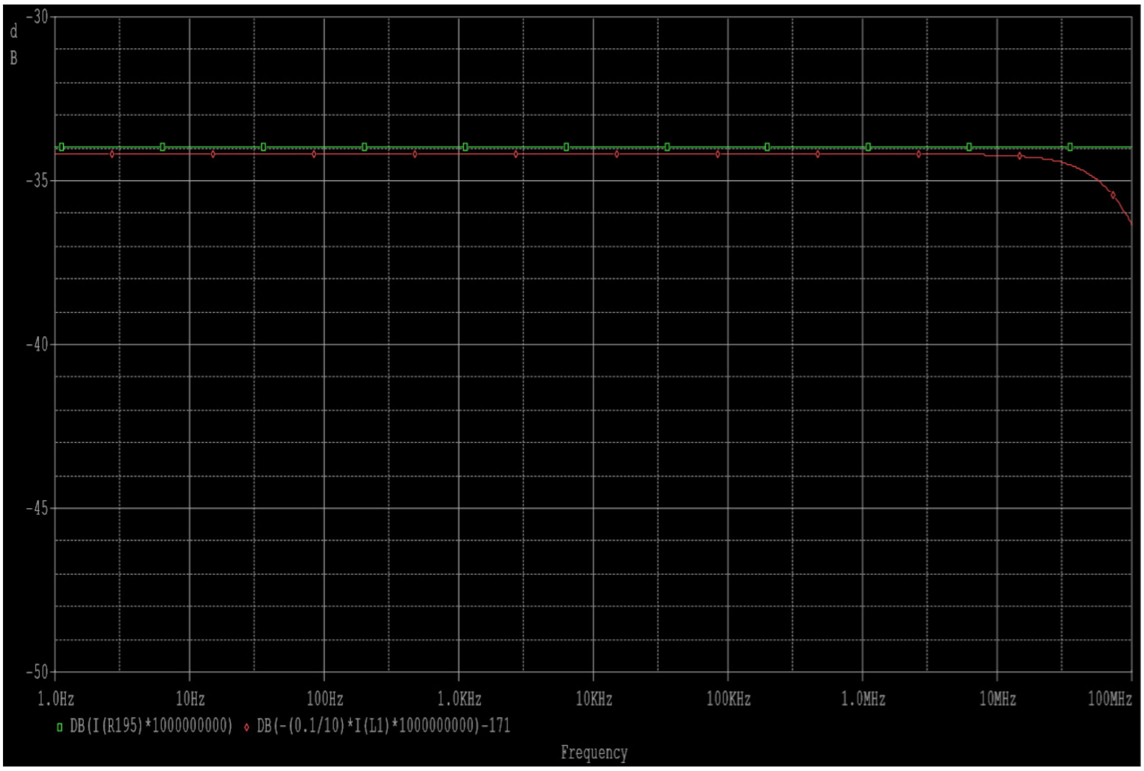

Fig.5: *Comparative frequency responses of V_{S1k}: Model based response ($\diamondsuit$), BSIM3v3 based benchmark ($\square$).*

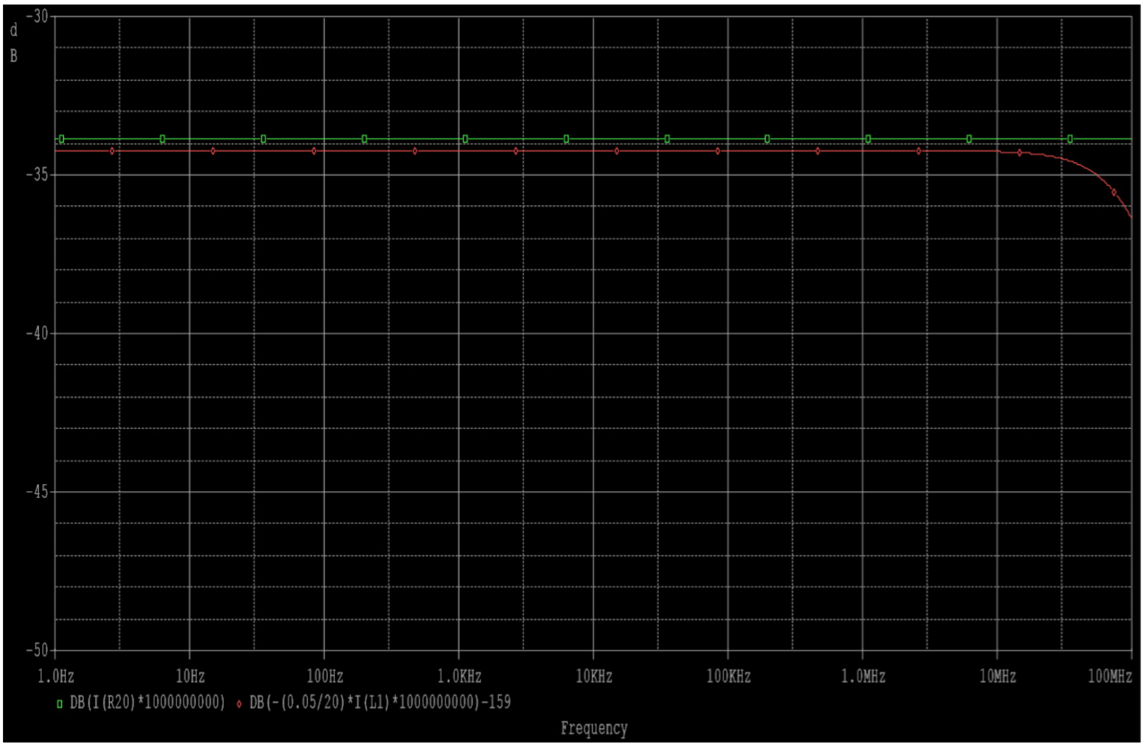

Fig.6: *Comparative frequency responses of V_{S2k}: Model based response ($\diamondsuit$), BSIM3v3 based benchmark ($\square$).*

depicted as in Fig. 7 where the dynamic transformer is composed of the stator coil and field coil which respectively serve as the secondary and primary winding. Since this transformer has bidirectional coupling, we have

$$Z_{11k}^{mSP} \neq 0 \tag{38}$$

$$Z_{11k}^{mPS} \neq 0 \tag{39}$$

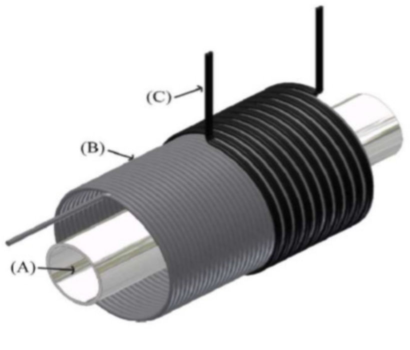

Fig.7: *The schematic diagram of FT-FCG (A) armature, (B) stator coil, (C) filed coil [16].*

As a result, we obtained

$$I_{P1k} = (Z_{11k}^{sP} + \frac{k_{11k}^{PS} Z_{11k}^{mPS}}{n_{11k}^{PS}})^{-1} V_{P1k} \tag{40}$$

$$I_{S1k} = \frac{k_{11k}^{sP}}{n_{11k}^{SP}} (Z_{11k}^{sP} + \frac{k_{11k}^{PS} Z_{11k}^{mPS}}{n_{11k}^{PS}})^{-1} V_{P1k} \tag{41}$$

$$V_{S1k} = \frac{k_{11k}^{sP}}{n_{11k}^{SP}} \frac{Z_{11k}^{sS} + \frac{k_{11k}^{SP} Z_{11k}^{mSP}}{n_{11k}^{SP}}}{Z_{11k}^{sP} + \frac{k_{11k}^{PS} Z_{11k}^{mPS}}{n_{11k}^{PS}}} V_{P1k} \tag{42}$$

The parameters of this transformer are time dependent, so it cannot be claimed that $I_{p1} = I_{p2} \ldots = I_{pK}$, $I_{s1} = I_{s2} \ldots = I_{SK}$, $V_{S1} = V_{S2} \ldots = V_{SK}$, $Z_{111}^{sP} = Z_{112}^{sP} \ldots = Z_{11K}^{sP}$, $Z_{111}^{sS} = Z_{112}^{sS} \ldots = Z_{11K}^{sS}$, $Z_{111}^{mSP} = Z_{112}^{mSP} \ldots = Z_{11K}^{mSP}$, $Z_{111}^{mPS} = Z_{112}^{mPS} \ldots = Z_{11K}^{mPS}$. However, since such parameters are frequency independent Z_{11k}^{sP} , Z_{11k}^{sS} , Z_{11k}^{mSP} and Z_{11k}^{mPS} are simply linear functions and can be given by

$$Z_{11k}^{sP} = sL_{11k}^{sP} + R_{11k}^{sP} \tag{43}$$

$$Z_{11k}^{sS} = sL_{11k}^{sS} + R_{11k}^{sS} \tag{44}$$

$$Z_{11k}^{mSP} = sL_{11k}^{mSP} + R_{11k}^{mSP} \tag{45}$$

$$Z_{11k}^{mPS} = sL_{11k}^{mPS} + R_{11k}^{mPS} \tag{46}$$

These equations are also in terms of (13)-(16) but with $Q_{sPk} = 1$, $Q_{sSk} = 1$, $Q_{mPSk} = 1$, $Q_{mSPk} = 1$, $R_{sPk} = 0$, $R_{sSk} = 0$, $R_{mPSk} = 0$ and $R_{mSPk} = 0$.

At this point, I_{P1k}, I_{S1k} and V_{S1k} can be numerically simulated against the frequency and k by using MATHEMATICA based on the time dependent parameter values [16] as depicted in Fig. 8-10 where the frequency is in terms of logarithmic value and ranged from 0 to 6. On the other hand, k is ranged from 1 to 16 as we let $K = 16$. Note also that the resulting values of I_{P1k}, I_{S1k} and V_{S1k} have been interpolated and the cosinusoidal waveform with unity magnitude has been adopted as the input voltage. Moreover, the adjacent time instances, e.g. $k = 1$ and $k = 2$ etc., are 10-6 sec apart. From these figures, in addition to the frequency dependencies, time dependencies of I_{P1k}, I_{S1k} and V_{S1k} can be observed and become strong when k approaches 16. This is because the time dependencies of the dynamic transformer's parameters are weak initially, but become stronger as time goes by [16]. Before we proceed further, it should be mentioned here that the previous tensor algebraic models, including that proposed in [24], cannot be conveniently applied in this scenario as they use tensors of order 1 for the current/voltage tensors and order 2 tensors for the impedance and current/voltage transformation tensors. As a result, the current/voltage functions become functions of both complex frequency and time. Moreover, the elements of the transformation tensors and the coefficients of the impedance functions become time dependent. Therefore the analysis can be cumbersome.

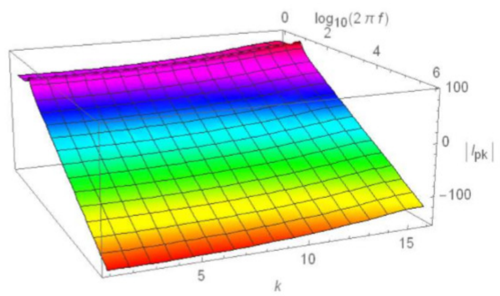

Fig.8: *Numerically simulated time variant frequency responses of I_{pk}.*

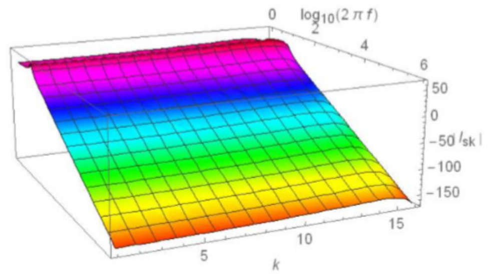

Fig.9: *Numerically simulated time variant frequency responses of I_{sk}.*

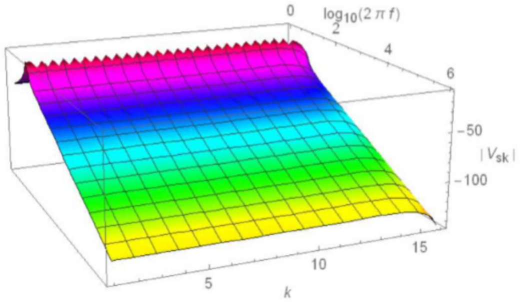

Fig.10: *Numerically simulated time variant frequency responses of* V_{sk}.

Finally, the fractional-order mutual inductance will be considered. Since this recent transformer has a single input and a single output, we have $N = 1$ and $M = 1$. Therefore primary current, secondary current, and secondary voltage can also be determined in terms of the primary voltage by using the proposed model as given by (40)-(42). Note also that $Z_{111}^{sP} = Z_{112}^{sP} \ldots = Z_{11K}^{sP}$, $Z_{111}^{sS} = Z_{112}^{sS} \ldots = Z_{11K}^{sS}$, $Z_{111}^{mSP} = Z_{112}^{mSP} \ldots = Z_{11K}^{mSP}$ and $Z_{111}^{mPS} = Z_{112}^{mPS} \ldots = Z_{11K}^{mPS}$, $I_{P1} = I_{P2} \ldots = I_{PK}$, $I_{S1} = I_{S2} \ldots = I_{SK}$ and $V_{S1} = V_{S2} \ldots = V_{SK}$ since the parameters of this transformer are time independent. By using the s-domain mathematical definition of the fractional impedances, Z_{11k}^{sP} , Z_{11k}^{sS} , Z_{11k}^{mSP} and Z_{11k}^{mPS} can be given as follows

$$Z_{11k}^{sP} = s^{\alpha} L_{11k}^{sP} \tag{47}$$

$$Z_{11k}^{sS} = s^{\beta} L_{11k}^{sS} \tag{48}$$

$$Z_{11k}^{mSP} = s^{\gamma SP} L_{11k}^{mSP} \tag{49}$$

$$Z_{11k}^{mPS} = s^{\gamma PS} L_{11k}^{mPS} \tag{50}$$

where α, β, γ_{SP} and γ_{PS} are not strictly integers but can be fractional according to [17]. Therefore Z_{11k}^{sP} , Z_{11k}^{sS} , Z_{11k}^{mSP} and Z_{11k}^{mPS} are fractional order s-domain polynomial functions.

If we let $\alpha{=}\beta{=}\gamma_{SP}{=}\gamma_{PS} = 0.5$, which implies that the fractional-order mutual inductance is symmetric and all impedances are of half order then Z_{11k}^{sP} , Z_{11k}^{sS} , Z_{11k}^{mSP} and Z_{11k}^{mPS} can be given in terms of the integer order rational polynomial functions by using the first order continued fraction expansion [29] as

$$Z_{11k}^{sP} = \frac{3L_{11k}^{sP} s + L_{11k}^{sP}}{s+3} \tag{51}$$

$$Z_{11k}^{sS} = \frac{3L_{11k}^{sS} s + L_{11k}^{sS}}{s+3} \tag{52}$$

$$Z_{11k}^{mSP} = \frac{3L_{11k}^{mSP} s + L_{11k}^{mSP}}{s+3} \tag{53}$$

$$Z_{11k}^{mPS} = \frac{3L_{11k}^{mPS} s + L_{11k}^{mPS}}{s+3} \tag{54}$$

If the fifth order continued fraction expansion has been adopted Z_{11k}^{sP} , Z_{11k}^{sS} , Z_{11k}^{mSP} , and Z_{11k}^{mPS} become

$$Z_{11k}^{sP} = \frac{11L_{11k}^{sP}s^5 + 165L_{11k}^{sP}s^4 + 462L_{11k}^{sP}s^3 + 330L_{11k}^{sP}s^2 + 55L_{11k}^{sP}s + L_{11k}^{sP}}{s^5+55s^4+330s^3+462s^2+165s+11} \tag{55}$$

$$Z_{11k}^{sS} = \frac{11L_{11k}^{sS}s^5 + 165L_{11k}^{sS}s^4 + 462L_{11k}^{sS}s^3 + 330L_{11k}^{sS}s^2 + 55L_{11k}^{sS}s + L_{11k}^{sS}}{s^5+55s^4+330s^3+462s^2+165s+11} \tag{56}$$

$$Z_{11k}^{mSP} = \frac{11L_{11k}^{mSP}s^5 + 165L_{11k}^{mSP}s^4 + 462L_{11k}^{mSP}s^3 + 330L_{11k}^{mSP}s^2 + 55L_{11k}^{mSP}s + L_{11k}^{mSP}}{s^5+55s^4+330s^3+462s^2+165s+11} \tag{57}$$

$$Z_{11k}^{mPS} = \frac{11L_{11k}^{mPS}s^5 + 165L_{11k}^{mPS}s^4 + 462L_{11k}^{mPS}s^3 + 330L_{11k}^{mPS}s^2 + 55L_{11k}^{mPS}s + L_{11k}^{mPS}}{s^5+55s^4+330s^3+462s^2+165s+11} \tag{58}$$

With the fifth order Oustaloup's approximation [28] Z_{11k}^{sP} , Z_{11k}^{sS} , Z_{11k}^{mSP} , and Z_{11k}^{mPS} can be now respectively given by (59)-(62). It can be seen that (51)-(62) are in terms of (13)-(16), with $Q_{sPk} = 1$, $Q_{sSk} = 1$, $Q_{mPSk} = 1$, $Q_{mSPk} = 1$, $R_{sPk} = 1$, $R_{sSk} = 1$, $R_{mPSk} = 1$ and $R_{mSPk} = 1$ for (51)-(54). $Q_{sPk} = 5$, $Q_{sSk} = 5$, $Q_{mPSk} = 5$, $Q_{mSPk} = 5$, $R_{sPk} = 5$, $R_{sSk} = 5$, $R_{mPSk} = 5$, and $R_{mSPk} = 5$ for (55)-(62)

$$Z_{11k}^{sP} = \frac{10L_{11k}^{sP}s^5 + 298.5L_{11k}^{sP}s^4 + 1218L_{11k}^{sP}s^3 + 768.75L_{11k}^{sP}s^2 + 74.97L_{11k}^{sP}s + L_{11k}^{sP}}{s^5+74.95s^4+768.5s^3+1218s^2+298.5s+10} \tag{59}$$

$$Z_{11k}^{sS} = \frac{10L_{11k}^{sS}s^5 + 298.5L_{11k}^{sS}s^4 + 1218L_{11k}^{sS}s^3 + 768.75L_{11k}^{sS}s^2 + 74.97L_{11k}^{sS}s + L_{11k}^{sS}}{s^5+74.95s^4+768.5s^3+1218s^2+298.5s+10} \tag{60}$$

$$Z_{11k}^{mSP} = \frac{10L_{11k}^{mSP}s^5 + 298.5L_{11k}^{mSP}s^4 + 1218L_{11k}^{mSP}s^3 + 768.75L_{11k}^{mSP}s^2 + 74.97L_{11k}^{mSP}s + L_{11k}^{mSP}}{s^5+74.95s^4+768.5s^3+1218s^2+298.5s+10} \tag{61}$$

$$Z_{11k}^{mPS} = \frac{\begin{array}{c}10L_{11k}^{mPS}s^5 + 298.5L_{11k}^{mPS}s^4 + 1218L_{11k}^{mPS}s^3 \\ + 768.75L_{11k}^{mPS}s^2 + 74.97L_{11k}^{mPS}s + L_{11k}^{mPS}\end{array}}{s^5+74.95s^4+768.5s^3+1218s^2+298.5s+10} \tag{62}$$

Let us assume that the fractional-order mutual inductance is connected to a 1 Ω resistive load where a cosinusoidal waveform with unity magnitude has been adopted as its input voltage. The frequency responses of the load voltage Vload can be numerically simulated by using our extensive tensor algebraic model based solutions, the first and fifth order continued fraction expansion, and the fifth order Oustaloup's approximation under the assumption that $\alpha = \beta = \gamma_{SP} = \gamma_{PS} = 0.5$ and $L_{11k}^{sP} = L_{11k}^{sS} = L_{11k}^{mSP} = L_{11k}^{mPS}$ μH [17] as shown in Fig. 11, where the frequency is in term of logarithmic value and ranged from 0 to 3. Moreover, the frequency responses of the half order fractional-order mutual inductance based V_{load} and the frequency responses of V_{load} calculated using the traditional matrix-vector linear algebraic approach have also been included.

From Fig. 11, it can be seen that the half order fractional-order mutual inductance based V_{load} displays a significantly different behaviour from that calculated using the traditional matrix-vector linear algebraic approach (fewer attenuation slopes) due to the influences of fractional impedances. This demonstrates the modelling failure of the traditional approach. On the other hand, our extensive tensor algebraic model based V_{load}'s are closer to the half order fractional-order mutual inductance based one which is our benchmark, particularly when the fifth order rational approximations have been applied. This verifies the advantages of our approach over the traditional approach.

It can also be seen that the error of the first order continued fraction expansion based V_{load} is higher than those of the fifth order continued fraction expansion based and fifth order Oustaloup's approximation based V_{load}'s. Since both fifth order rational approximations based V_{load}'s are totally coincident, as can be seen from the overlap of the red and magenta lines in Fig. 11, they have equal amounts of error. Therefore it can be stated that the error of the model based V_{load} from its half order fractional-order mutual inductance based benchmark is induced by the rational approximation error of $s^{0.5}$ and such error can be reduced by using higher order rational approximations despite their higher computational complexities. This is not surprising, because a higher order approximation gives a better result for a larger frequency range [29]. Therefore stronger agreement between our model based V_{load} and its benchmark can be expected if higher order approximations are used. Moreover, different approximation methodologies of equal orders yield insignificantly different results, unlike those of different orders.

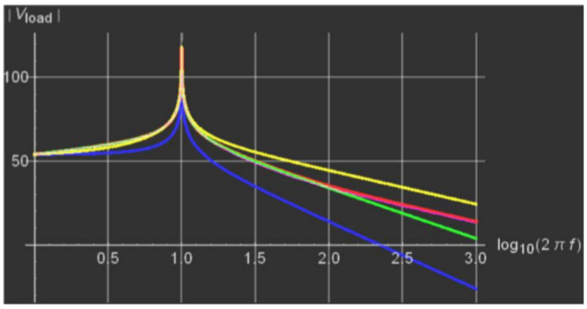

Fig.11: *Comparative frequency responses of Vload: Half order fractional-order mutual inductance based response (Green), Traditional matrix-vector linear algebraic approach based response (Blue), The first order continued fraction expansion/extensive tensor algebraic model based response (Yellow), The fifth order continued fraction expansion/ extensive tensor algebraic model based response (Magenta), The fifth order Oustaloup's approximation/extensive tensor algebraic model based response (Red).*

5. DISCUSSION

In this section, the effects of the failure of Kron's postulate on the power invariant and the validity of duality invariant will be discussed. After determining all of the transformer's current and voltage tensors, the electrical powers at all primary and secondary windings can be obtained. The arrays of powers at primary and secondary windings, i.e. $\boldsymbol{p_{Pik}}$ and $\boldsymbol{p_{Sjk}}$, are order 2 tensors which can be obtained by using the determined currents and voltage tensors as

$$\boldsymbol{p_{Pik}} = L^{-1}[\boldsymbol{V_{Pik}}] \otimes L^{-1}[\boldsymbol{I_{Pik}}] \tag{63}$$

$$\boldsymbol{p_{Sjk}} = L^{-1}[\boldsymbol{V_{Sjk}}] \otimes L^{-1}[\boldsymbol{I_{Sjk}}] \tag{64}$$

where $L^{-1}[\ ]$ and $\otimes$ stand for the inverse Laplace transform operator and the Hadamard product operator respectively.

It can be seen from (63) and (64) that each element of $\boldsymbol{p_{Pik}}$ and $\boldsymbol{p_{Sjk}}$, p_{Pik} and p_{Sjk}, can be respectively given as follows

$$p_{Pik} = L^{-1}[V_{Pik}]L^{-1}[I_{Pik}] \tag{65}$$

$$p_{Sjk} = L^{-1}[V_{Sjk}]L^{-1}[I_{Sjk}] \tag{66}$$

Therefore, by the failure of Kron's postulate on power invariant [22] and the convolution theorem, we have found that the voltage-current relationship given by (67), where * stands for the convolution in s-domain is invalid.

$$\forall k[\sum_{i=1}^{N}[V_{Pik} * I_{Pik}] = \sum_{j=1}^{M}[V_{Sjk} * I_{Sjk}]] \tag{67}$$

However, because of the validity of the duality invariant [22], the following voltage-current relationship is always valid

$$\begin{aligned}\forall k[\sum_{i=1}^{N}[V_{Pik} * I_{Pik}] + \sum_{i=1}^{N}[V_{PDik} * I_{PDik}] \\ = \sum_{j=1}^{M}[V_{Sjk} * I_{Sjk}] + \sum_{j=1}^{M}[V_{SDjk} * I_{SDjk}]]\end{aligned} \tag{68}$$

where V_{PDik} , I_{PDik} , V_{SDik} , and I_{SDik} denote the primary voltage, primary current, secondary voltage, and secondary current of arbitrary $\mathrm{i^{th}}$ primary and $\mathrm{j^{th}}$ secondary winding at an arbitrary $\mathrm{k^{th}}$ instant of the dual circuit of transformer of interest. Such a dual circuit is also a transformer, but with specifically altered configuration, e.g. interchanged primary and secondary windings [30].

6. CONCLUSIONS

This research proposes an extensive s-domain tensor algebraic model that is applicable to recent transformers which employ the unconventional characteristics. Unlike [19], [20] and [24], the impedance and current/voltage transformation tensors are of order 3 and the current/voltage tensors are of order 2. Because of this and other features, our proposed model is more inclusive than the traditional matrixvector algebraic approaches and previous tensor algebraic models. It is more complicated because tensors with higher order have been assumed. It can be applied to transformers with unconventional characteristics with higher accuracy, but more computational effort is required. Applications of the proposed model to typical recent transformers with unconventional characteristics, namely CMOS gyrator-C based active transformer, the dynamic transformer for FT-FCG, and the fractional-order mutual inductance, have been shown. A discussion on the effects of the failure of Kron's postulate and the validity of duality invariant to voltage-current relationships of the transformer has also been given. Therefore this work is beneficial to the analysis and design of circuits and systems involving recent transformers in practice. Compared to the conventional matrix-vector approach and previous tensor algebraic approaches, our extensive tensor algebraic modelling is more efficient and up to date for the mathematical modelling of recent transformers.

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