An Optimal Integer Partition Approach to Coalition Structure Generation

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ABSTRACT

This paper proposes a new solution to the problem of coalition structure generation using an optimal integer partition. The new partition is the set of set of integers where each integer represents size of a coalition. It includes only elements that no other elements in this partition have values of generated structures higher than them. We show that an element of this partition is a set containing 1 at most one element. Any solutions to the problem of coalition structure generation using the new partition can reduce at least approximately 40% of possible candidate structures when size of coalition members at least 5. Moreover, the bigger the size of coalition members is, the more these solutions outperform the ones using integer partitions.

Keywords: Integer Partition, Coalition Structure, Coalition Formation, GroupBuying, E-Marketplaces

1. INTRODUCTION

Coalition structure generation is an activity of coalition formation. Its purpose is to find optimal coalition structures. The problem of searching for such optimal structures is computationally complex and shown to be NP-hard because the size of different possible coalition structures is exponential in the number of coalition members. This prohibits enumeration of all coalition structures and evaluation of them to find optimal coalitions. Therefore, existing solutions [e.g., 1, 2, 3] to this problem is to enumerate some candidate structures and pick the best seen so far. The order-based genetic algorithm (OBGA), developed by Sen and Dutta [2], enumerates only promising structures using genetic algorithms. But it does not guarantee finding the optimal coalition structures. Sandholm et al. [1] proposed a minimal search algorithm for finding solutions within a bound from the optimal. Dang and Jennings [3] improved the algorithm of Sandholm et al. [1] on searching fewer coalition structures. However, all of these solutions are based on a huge space of entire structures - even some of these structures are not necessarily considered to be the optimal. As such, a search space should include only candidate structures proving to be optimal. We, therefore, propose to provide such a minimal space using an optimal integer partition.

The remainder of the paper is organized as follows. Section 2 gives background, and we present an optimal integer partition approach to the problem of coalition structure generation in section 3. Finally, conclusion is given in section 4.

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2. BACKGROUND

2.1 Basic Definitions

Definition 1 Let A be a finite set containing at least two elements and M be a power set of A. A set of coalition of A, called C, is a set of M - $\{\phi\}$

Example 1 Suppose $A = \{b_1, b_2, b_3, b_4\}$.The set of coalition of A is

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$\{b_1\}$,$\{b_2\}$,$\{b_3\}$,$\{b_4\}$,$\{b_1, b_2\}$,$\{b_1, b_3\}$,$\{b_1, b_4\}$,
$\{b_2, b_3\}$,$\{b_2, b_4\}$,$\{b_3, b_4\}$,$\{b_1, b_2, b_3\}$,
$\{b_1, b_2, b_4\}$,$\{b_1, b_3, b_4\}$,$\{b_2, b_3, b_4\}$,
$\{b_1, b_2, b_3, b_4\}$,
}.

Definition 2 Given$A= \{b_1, b_2, ..., b_n\}$,let Cbe the set of coalition of A and S be an element of C.A coalition structure of A,denoted as CS_A,is$\{CT_1, CT_2, ..., CT_j\}$;

where$\{CT_1\} = \{S_i | \, S_i \in C, \bigcap_{i=1}^{m} S_i = \phi, \bigcup_{i=1}^{m} S_i = A$,and m is the number of groups$\}$,for all k$\in \{1, 2, ..., j\}$.

Example 2 Given$A = \{b_1, b_2, b_3, b_4\}$and let C be the set of coalition of A. Therefore, the set of coalition structures of A is

{
$\{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}\}, \{\{b_1\}, \{b_2\}, \{b_3, b_4\}\}$,
$\{\{b_1\}, \{b_3\}, \{b_2, b_4\}\}, \{\{b_1\}, \{b_4\}, \{b_2, b_3\}\}$,
$\{\{b_2\}, \{b_3\}, \{b_1, b_4\}\}, \{\{b_2\}, \{b_4\}, \{b_1, b_3\}\}$,
$\{\{b_3\}, \{b_4\}, \{b_1, b_2\}\}, \{\{b_1\}, \{b_2, b_3, b_4\}\}$,
$\{\{b_2\}, \{b_1, b_3, b_4\}\}, \{\{b_3\}, \{b_1, b_2, b_4\}\}$,
$\{\{b_4\}, \{b_1, b_2, b_3\}\}, \{\{b_1, b_2\}, \{b_3, b_4\}\}$,
$\{\{b_1, b_3\}, \{b_2, b_4\}\}, \{\{b_1, b_4\}, \{b_2, b_3\}\}$,
$\{b_1, b_2, b_3, b_4\}$,
}.

Definition 3 Given $A = \{b_1, b_2, ..., b_n\}$,let C be the set of coalition of A and S be an element of C.The value of each coalition$S$, denoted as *v(S)*,is 0 when $|S|$=1 and greater than 0 otherwise.

Definition 4 Given $A = \{b_1, b_2, ..., b_n\}$and let C be the set of coalition of A. The value of coalition structure CT, denoted as *V(CT)*, is $\sum_{S \in CT} v(S)$ where $CT \in CS_A$

Example 3 Suppose $CT = \{\{b_1, b_2, b_3, b_4\}\}$.Then $V(CT) = (\{b_1\}) + (\{b_2\}) + (\{b_3\}) + (\{b_4\}) = 0$

2.2 Number of Coalition Structures

The Stirling number of the second kind [4] is the number of ways of partitioning a set of n elements into non-empty k elements set, denoted as *S(n, k)*. This number can be calculated using the following formula.

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n$$

Example 4 Suppose $A = \{b_1, b_2, b_3, b_4\}$.S(4,k),where k = 1, 2, 3, 4, can be calculated using (1) as followed.

S(4,4)= 1: $\{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}\}$
S(4,3)= 6: $\{\{b_1\}, \{b_2\}, \{b_3, b_4\}\}$,
$\{\{b_1\}, \{b_3\}, \{b_2, b_4\}\}$,
$\{\{b_1\}, \{b_4\}, \{b_2, b_3\}\}$,
$\{\{b_2\}, \{b_3\}, \{b_1, b_4\}\}$,
$\{\{b_2\}, \{b_4\}, \{b_1, b_3\}\}$,
$\{\{b_3\}, \{b_4\}, \{b_1, b_2\}\}$,
S(4,2)= 7: $\{\{b_1, b_2\}, \{b_3, b_4\}\}$,
$\{\{b_1, b_3\}, \{b_2, b_4\}\}$,
$\{\{b_1, b_4\}, \{b_2, b_3\}\}$,
$\{\{b_1\}, \{b_2, b_3, b_4\}\}$,
$\{\{b_2\}, \{b_1, b_3, b_4\}\}$,
$\{\{b_3\}, \{b_1, b_2, b_4\}\}$,
$\{\{b_4\}, \{b_1, b_2, b_3\}\}$,
S(4,4)= 1: $\{\{b_1, b_2, b_3, b_4\}\}$

The sum $\sum_{k=1}^{n} S(n,k)$is the number of coalition structures of n members into the non-empty k elements set.

Example 5 $\sum_{k=1}^{4} S(n,k) = S(4,1) + S(4,2) + S(4,3) + S(4,4)$ = 1+6+7+1= 15. Therefore, the number of coalition structures for a set with 4 members is 15.

The triangle of Stirling numbers of the second kind is shown in figure 1. The sum of numbers in the ith row is the number of coalition structures of a set with i elements. Sandholm et. al. [1] show that the number of coalition structures of a set with n members is $O(n^n)$.

1
1 1
1 3 1
1 7 6 1
1 15 25 10 1
1 31 90 65 15 1
1 63 301 350 140 21 1
1 127 966 1701 1050 266 28 1
1 255 3025 7770 6951 2646 462 36 1

Fig.1: *The Triangle of Stirling Numbers of the Second Kind*

3. AN OPTIMAL INTEGER PARTITION APPROACH

3.1 The Problem of Coalition Structure Generation

The optimal coalition structure is the structure $CT^* = \mathbf{arg\ max}_{CT \in CS} V(CT)$ .The problem of coalition structure generation is to search for such a coalition structure. It is NP-hard and any solutions to it must search an exponential number of candidate structures [1]. However, some candidate structures are not necessarily considered to be optimal coalition structures, because their values equal 0 or less than values of other candidate structures. Consider example 4 and assume that a, b, c, and d are positive real numbers. From definition 3, we have

$$v(\{b_1\}) = v(\{b_2\}) = v(\{b_3\}) = v(\{b_4\}) = 0$$

With the same definition, also suppose that

$$v(\{b_1, b_2\}) = a,$$
$$v(\{b_3, b_4\}) = b,$$
$$v(\{b_1, b_2, b_3\}) = c,$$
$$\text{and } v(\{b_1, b_2, b_3, b_4\}) = d.$$

Therefore, we have

$$V(\{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}\}) = 0 + 0 + 0 + 0,$$
$$V(\{\{b_3\}, \{b_4\}, \{b_1, b_2\}\}) = 0 + 0 + a,$$
$$V(\{\{b_1\}, \{b_2\}, \{b_3, b_4\}\}) = 0 + 0 + b,$$
$$V(\{\{b_1, b_2\}, \{b_3, b_4\}\}) = a + b,$$
$$V(\{\{b_1, b_2, b_3\}, \{b_4\}\}) = c + 0,$$
$$\text{and } V(\{\{b_1, b_2, b_3, b_4\}\}) = d + 0,$$

Since $V(\{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}\}) = 0, V(\{\{b_3\}, \{b_4\}, \{b_1, b_2\}\}) < V(\{\{b_1, b_2\}, \{b_3, b_4\}\})$, and $V(\{\{b_1\}, \{b_2\}, \{b_3, b_4\}\}) < V(\{\{b_1, b_2\}, \{b_3, b_4\}\})$, we dont need to include $(\{\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}\})$,$(\{\{b_3\}, \{b_4\}, \{b_1, b_2\}\})$and $V(\{\{b_1\}, \{b_2\}, \{b_3, b_4\}\})$as candidate structures. However, $V(\{\{b_1, b_2, b_3\}, \{b_4\}\})$ and V($\{b_1, b_2, b_3, b_4\}$)can be optimal structures because their values are neither 0 nor less than values of other candidate structures. This observation combined with integer partition gives an optimal integer partition as shown in the next section.

3.2 An Optimal Integer Partition

Definition 5 Let n be a positive integer.Partition of n ,denoted as P_n, is $\{L_1, L_2, L_{...}L_p\}$; where $L_{k\in\{1,2,...,p\}}$

$$\{a_i| \sum_{i=1}^{m} = \text{n, } a_i \text{ is a positive integer,}$$

$$\text{and m is number of groups}\}.$$

Definition 6 Coalition structure of partition of n, denoted as CS_{P_n} , $is \bigcup_{x=1}^{|P_n|} CT_{L_x}$ where $CT_{L_x} = \{CP_1, CP_2, ..., CP_y\}$ and CP_Z is $\{S_i| \bigcup_{i=1}^{m} S_i = \phi, \bigcup_{i=1}^{m} S_i = A, |S_i| = a_i$ and $m = |L_x|$ }.

Example 6 Suppose $A = \{b_1, b_2, b_3, b_4\}$and let C be the set of coalition of A . From definition 5, we have $P_4(\{\{4\}, \{3,1\}, \{2,2\}, \{2,1,1\}, \{1,1,1,1\}\})$.$CS_{P_4}$ can be determined according to definition 6 as shown in table 1.

Table 1: CS_{P_4}

L	CT_L
{4}	{ {b1,b2,b3,b4} }
{3,1}	{ {{b1,b2,b3},{b4}}, {b1,b2,b4},{b3}}, {{b1,b3,b4},{b2}}, {{b2,b3,b4},{b1}}}
{2,2}	{ {{b1,b2},{b3,b4}}, {b1,b3},{b2,b4}}, {{b1,b4},{b2,b3}}}
{2,1,1}	{ {{b1,b2},{b3},{b4}}, {{b1,b3},{b2},{b4}}, {{b1,b4},{b2},{b3}}, {{b2,b3},{b1},{b4}}, {{b2,b4},{b1},{b3}}, {{b3,b4},{b1},{b2}} }
{1,1,1,1}	{ {{b1},{b2},{b3},{b4}} }

Definition 7 An optimal integer partition of n, denoted as P_n^*, is $\{L_t| L_t \in P_n$, not exist $L_h \in P_n$ that $V(CTL_h) > V(CTL_t)$ and $t \neq h$ }

Observation Consider table 1 and definition 7, values of any coalition structures generated from L containing 1 at least two elements are always less than values of some coalition structures generated from L containing 1 at most one element. Therefore, P_n^* contains only L with 1 at most one element, as shown in theorem 1.

Theorem 1, If $L_k = \{a_1, a_2, ..., a_x, 1_1, 1_2, ..., 1_y\}$ and ≥ 2, then $L_k \notin P_n^*$

Proof Assume $L_k = \{a_1, a_2, ..., a_x, 1_1, 1_2, ..., 1_y\}$ and ≥ 2.We can find

$L^* = \{a_1, a_2, ..., a_x, a_{x+1}, 1_1, 1_2, ..., 1_y\}$,

where $r = n - \{a_1 + a_2 + ..., a_x, a_{x+1}\}$.

From definition 2 and 6, we have

$V(CT_{L_k}) = v(a_1) + v(a_2) + ... + v(a_x)$,

and $V(CT_{L^*}) = v(a_1) + v(a_2) + ... + v(a_x + v(a_{x+1})$.

Hence, $V(CT_{L^*}) > V(CT_{L_k})$.

Therefore, $L_k \notin P_n^*$

For any $L_k \notin P_n$

$$L = \left\{ \underbrace{1, 1, ...}_{m_1 terms}, \underbrace{2, 2, ...}_{m_2 terms}, \underbrace{3, 3, ...}_{m_3 terms}, ... \right\}.$$

the number of coalition structures ($|CT_L|$)is the Fa di Bruno coefficient [5]

$$\frac{n!}{(m!_1 m!_2 m!_3)(1!^{m_1} 2!^{m_2} 3!^{m_3} ...)}$$

Therefore, we have

$$|CS_{P_n}| = \sum_{k=1}^{|P_n|} CT_{L_k}$$

and

$$|CS_{P_n^*}| = \sum_{k=1}^{|P_n^*|} CT_{L_k}$$

Table 2 shows$|CS_{P_n}|$,$|CS_{P_n^*}|$,and reduction percentage at n = 5, 10, 20, 30, 40, and 50. The results show that (1) new partitions can reduce at least approximately 40% of possible candidate structures and (2) the bigger n is, the more new partitions outperforms integer partitions.

4. CONCLUSION

This paper proposes to reduce a search space of the problem of coalition structure generation using a new optimal integer partition. The new partition includes

Table 2: *Reduction of Structures*

n	$\left\|CS_{P_n}\right\|$	$\left\|CS_{P_n^*}\right\|$	Reduction (%)
5	52	31	40.38
10	115975	51972	55.19
20	5.1724×10^{13}	1.74×10^{13}	66.36
30	8.4675×10^{23}	2.35×10^{23}	72.25
40	1.5745×10^{35}	3.79×10^{34}	75.93
50	1.8572×10^{47}	3.98×10^{46}	78.57

only elements that no other elements in this partition have values of generated structures higher than them. This directly reduces number of candidate structures without losing complete optimal structures. We show that an element of the partition is a set containing 1 at most one element. Any solutions to the problem of coalition structure generation using this new partition can reduce a search space approximately at least 40% and the bigger the size of coalition members is, the more the solutions outperform the ones using integer partitions.

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