An Analysis of a Close-Type Queuing System with General Service-Time Distribution

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ABSTRACT

This paper presents an innovative approach to solve probability distributions of a close feed back loop type queuing system with general service time distribution. This model is applied to a multi-processor system where some of its nodes are performed a repair procedure during a nodes malfunction condition. Our model is appropriate for a multiprocessor system that employs a common bus or for a multi-node system in computer networks. A meticulous analysis of the systems model has been conducted and numerical results have been obtained to scrutinize the proposed model.

Keywords: Closed queuing system, general service distribution, fault tolerance

1. INTRODUCTION

The queuing system is widely classified into an open-type system and a closed-type system. In the open-type system model, customers arrive from outside and depart to the outside of the system while in the closed system, the customers operate internally where no customers arrive from outside or depart to outside of the system.

Numerous research works have been extensively dedicated to investigate the open system model which is widely used in computer systems and computer networks [1]-[6]. However, the closed system model has not been paid much attention in spite of its paramount importance to computer systems [7]. This paper has made a very punctilious effort to formulate the closed systems behavior in the course of malfunctions which might develop in the system during repairing procedures.

In the analytical model of this paper, we define λ as the failure rate (or arrival rate) and μ as the rate of completion of the repairing procedure (or service rate) when the system is busy. Meanwhile, the exponential distributions for both arrival time and service time have been considered.

A multi-processor system that is connected to a central bus controller or an arbiter was selected as an example in our proposed model. In above descriptions, there are many closed type systems in the computer systems.

This paper presents the analytical method of the closed feed back loop queuing model with a general service distribution. Furthermore, numerical examples of the close-type queuing model are given and the system performances are also discussed.

2. MODEL OF A CLOSED LOOP TYPE QUEUING SYSTEM

The proposed system consists of multi elements or processors which are autonomously operated. If any malfunctioning element of the system is detected then this element is required for repair operation (or service) at the service repair center (server). The repaired element is subsequently put into service again.

This kind of elements failure and repairing procedure in a close system environment is called Closed Feed Back Loop Type *CFBLT* queuing system. An example of this model is illustrated in Fig. 1.

The *CFBLT* model has fault tolerance with respect to a single or a multi-element failure. The system has also self configuration feature that can tolerant temporary failures while tasks which have been assigned to the failure element are distributed to other active elements. The system can tolerant up to m of $N+1$ elements ($m \in N+1$) while the system operation will be normal operation condition if the number of faulty elements are less than or equal to m.

As mentioned above, the proposed systems model is applicable for numerous systems applications in computer modeling.

3. AN ANALYSIS OF THE CLOSED QUEUING SYSTEM

A systematic approach is given in this section for proper analysis of the closed feed back loop type queuing system, an example of this model is shown in Fig. 1.

The *CFBLT* model is described below.

(1) We consider the system is in a steady state, and the queue is first in and first service (FIFS) models discipline.

(2) Let the number of elements in the system be $N+1$ ($N>0$). The request arrivals for service due to elements malfunctions follow an exponential distribution with failure rate (arrival rate) of one element λ.

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(2) Let the service-time distribution be a general distribution. Suppose the probability that a service is started between arbitrary time t and time t+Δ is equal to $\mu(x)\Delta+o(\Delta)^2$,and the density function *f(x)*

$$f(x)=\mu(x).e^{-\int_0^x \mu(y)dy} \quad (1)$$

(4) Let the probability density function of the service-time x with n queue length be $w_n(x)$. The state probabilities p_{n+1} are given by

of service-time distribution is given by

$$p_{n+1}=\int_0^\infty w_n(x)dx(n=0,1,2,...,N) \quad (2)$$

From above notations, since the service is continued as shown in Fig. 2, the relationship between $w_n(x)$ in arbitrary time t and $w_n(x+\Delta)$ in time $(t+\Delta)$, as shown in the left side of Fig. 2, is given as follows

$$wo(x+\Delta)$$

$$=\{1-(N-n)\lambda\Delta\}\{1-\mu(x)\Delta\}w_0(x)+o(\Delta)^2 \quad (3)$$

$$w_n(x+\Delta)=\{1-(N-n)\lambda\Delta\}\{1-\mu(x)\Delta\}$$

$$w_n(x)+(N-n+1)\lambda\Delta w_{n-1}(x)+o(\Delta^2) \quad (4)$$

(n=1,2,...N)

for $\Delta\to 0$, the above equations become as follows

$$\frac{d}{dx}w_0(x)+\{N\lambda+\mu(x)\}w_0(x)=0 \quad (5)$$

$$\frac{d}{dx}w_n(x)+\{(N-n)\lambda+\mu(x)\}w_n(x)=0 \quad (6)$$

$$=(N-n+1)\lambda w_{n-1}(x) \quad (n=1,2,...N)$$

In order to solve the above differential equations, suppose $w_n(x)$ is define by the following general function

$$w_n(x)=h_n(x).e^{-(N-n)\lambda x-\int_0^x \mu(x)dx}(n=1,2,...,N) \quad (7)$$

where $h_n(x)$ is any function then the differential equations (5) and (6) become

$$\frac{d}{dx}h_0(x)=0 \quad (8)$$

$$=\frac{d}{dx}h_n(x)=(N-n+1)\lambda e^{-\lambda x}h_{n-1}(x) \quad (9)$$

(n=1,2,...,N)

From (8), $h_0(x)=C_0$, where C_0 is a constant value, and we will have the following solutions

$$h_n(x)=C_n-C_{n-1}\left\{\frac{N-n+1}{1!}\right\}e^{-\lambda x}$$

$$+C_{n-2}\left\{\frac{(N-n+2)(N-n+1)}{2!}\right\}e^{-2\lambda x}$$

$$+...+(-1)^n C_0\left\{\frac{N(N-1)...(N-n+1)}{2!}\right\}e^{-n\lambda x}$$

$$=C_n+\sum_{i=1}^{n}(-1)^i C_{n-i}\left\{\frac{(N-n+i)...(N-n+1)}{i!}\right\}e^{-i\lambda x}$$

(n=1,2,...,N) (10)

where $C_i(i=0,1,2,...,N)$ is a constant value given by the boundary conditions at the start point or at the end point of service. Moreover, from (2), the state probabilities are

$$p_1=C_0\left\{\frac{1-f^*(N\lambda)}{N\lambda}\right\} \quad (11)$$

$$p_{n+1}=C_n\left\{\frac{1-f^*((N-n)\lambda)}{(N-n)\lambda}\right\}$$

$$+\sum_{i=1}^{n}(-1)^i C_{n-i}\left\{\frac{(N-n+i)...(N-n+1)}{i!}\right\} \quad (12)$$

$$\left\{\frac{1-f^*((N-n+i)\lambda)}{(N-n+i)\lambda}\right\}...(n=1,2,...,N-1)$$

$$p_{N+1}=C_N T_S+\sum_{i=1}^{N}(-1)^i C_{N-i}\left\{\frac{1-f^*(i\lambda}{i\lambda}\right\} \quad (13)$$

where $f^*(i\lambda)$ is the Laplace transform of the function $f(x)$ and is given by

$$f^*=(i\lambda)=\int_0^\infty f(x)e^{-i\lambda x}dx \quad (n=1,2,...,N-1) \quad (14)$$

and T_S is the average service time given by

$$T_S=\frac{1}{\mu}=\int_0^\infty xf(x)dx \quad (15)$$

where μ is the mean service rate of general service distribution.

On the other hand, the boundary conditions at the start point or at the end point of the service, as shown in right side of Fig. 2, are given by the following formulas

$$p_0=\{1-(N+1)\lambda\Delta\}p_0$$

$$+\int_0^\infty \mu(x)\Delta w_0(x)dx+o(\Delta^2) \quad (16)$$

$$w_0(0)\Delta=\int_0^\infty \mu(x)\Delta w_1(x)dx+(N+1)\lambda\Delta p_0+o(\Delta^2) \quad (17)$$

$$w_{n-1}(0)\Delta=\int_0^\infty \mu(x)\Delta w_n(x)dx+o(\Delta^2) \quad (18)$$

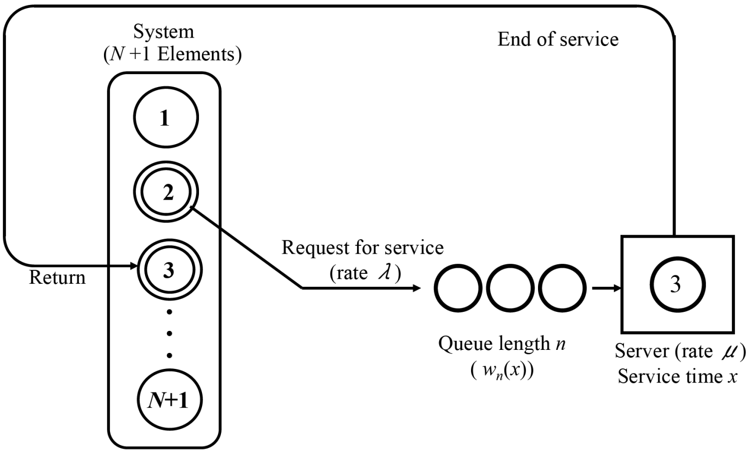

Fig.1: *An example model of a closed feed back loop type queuing system model*

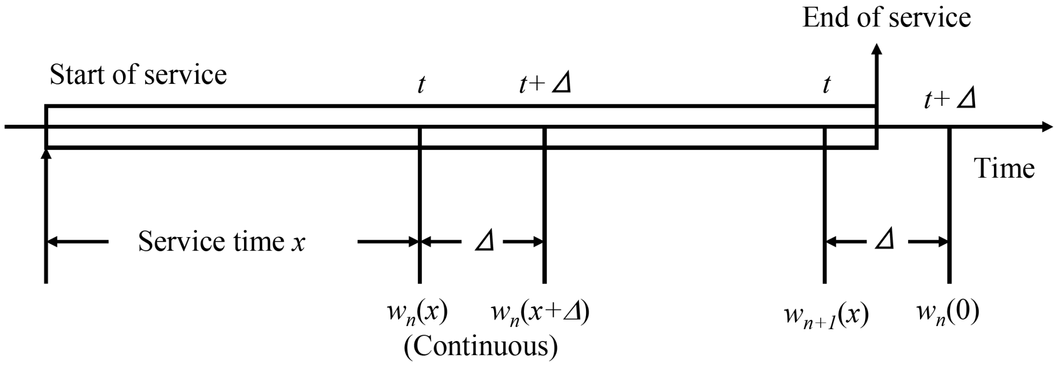

Fig.2: *The relationship among n states at arbitrary time t and t+Δ*

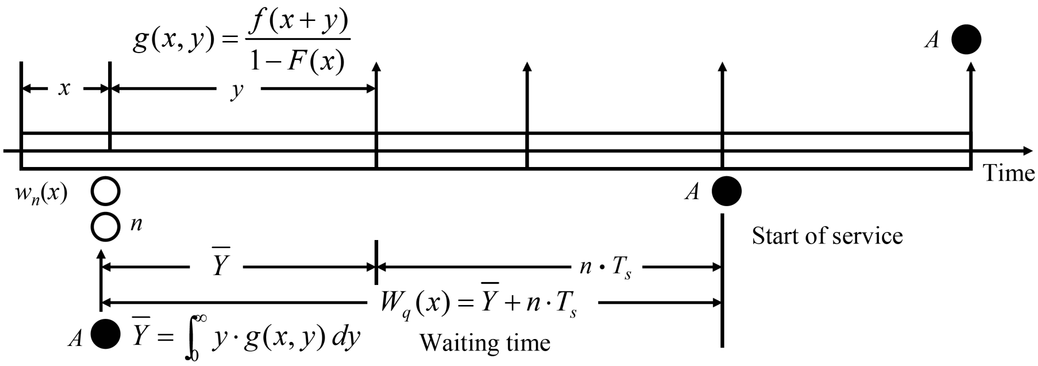

Fig.3: *Waiting time calculation when a new failure element A arrives at time x and with n-element in the queue*

$$(n = 1, 2, ..., N)$$

If $\Delta \to 0$, the above equations become as follows

$$(N+1)\lambda p_0 = \int_0^\infty \mu(x) w_0(x) dx \tag{19}$$

$$w_0(0) = \int_0^\infty \mu(x) w_1(x) dx + (N+1)\lambda p_0 \tag{20}$$

$$w_{n-1}(0) = \int_0^\infty \mu(x) w_n(x) dx \tag{21}$$

(n=1,2,...N)

Using the general solutions of (7) to (10), we are able to define the constant values of C_0, C_1 and C_n which are given by the following formulas

$$C_0 = \frac{(N+1)\lambda p_0}{f^*(N\lambda)} \tag{22}$$

$$C_1 = C_0 \left\{ \frac{1+(N-1)f^*(N\lambda)}{f^*((N-1)\lambda)} \right\} \tag{23}$$

$$C_n = \sum_{i=1}^{n-1} (-1)^i C_{n-i-1} \left\{ \frac{(N-n+i+1)\ldots(N-n+1)}{(i+1)!} \right\}$$

$$\left\{\frac{i+1+(N-n+1)f^*((N-n+i+1)\lambda)}{(N-n+1)f^*((N-n)\lambda)}\right\}$$

$$(n=1,2,..,N) \tag{24}$$

If we substitute the above constant values into the equations (11) to (13) then we have the state probabilities $p_{n+1}(n=0,1,2,...,N)$. The empty state probability p_0 is given by

$$\sum_{n=0}^{N+1} p_n = 1 \tag{25}$$

The state probabilities $p_n(n=0,1,2,...,N+1)$ are solved by above (11) to (13) and (25). Accordingly, the average queue length L_q and the average system length L are given respectively by

$$L_q = \sum_{n=1}^{N+1} (n-1)p_n = \sum_{n=0}^{N} p_{n+1} \tag{26}$$

$$L = \sum_{n=1}^{N+1} np_n = \sum_{n=1}^{N+1} (n-1)p_n + \sum_{n=1}^{N+1} p_n = Lq+ (1-p_0) \tag{27}$$

Now, we need to find out the average waiting time W_q of a new failure arrival. Fig. 3 describes how to get the waiting time $W_q(x)$, suppose a new failure element arrives at service time x, e.g. element A, on condition that n elements are waiting in a queue.

The waiting time of element A is $W_q(x)$, as shown in Fig. 3, and Wq(x) is given by

$$\begin{aligned} W_q(x) &= \bar{Y} + n\cdot T_S \\ &= \int_0^\infty y\cdot g(x,y)dy + n\cdot T_S \end{aligned} \tag{28}$$

where T_s is mean time of the service, $\bar{Y}$ is mean time of remaining service time and g(x, y) is conditional probability density function and is given by

$$g(x,y) = \frac{f(x+y)}{1-F(x)} \tag{29}$$

In (29), *f(x)* and *F(x)* are probability density function and probability distribution function of the service time, respectively. Then, the mean waiting time W_q of element A, that is given in (28), is calculated as follows

$$W_q = \sum_{n=0}^{N} \int_0^\infty w_n(x)\cdot Wq(x)dx$$

$$= \sum_{n=0}^{N} \int_0^\infty w_n(x)\cdot \left\{\int_0^\infty y\cdot g(x,y)dy + n\cdot T_S\right\}$$

$$= \sum_{n=0}^{N} \int_0^\infty w_n(x)\cdot \left\{\int_0^\infty y\cdot \frac{f(x+y)}{1-F(x)}dy + n\cdot T_S\right\} dx$$

$$= L_qT_S +$$

$$\sum_{n=0}^{N}\left\{C_n \frac{f^*((N-n)\lambda)}{((N-n)\lambda)^2} + \sum_{i=1}^{n} (-1)^i C_{n-i} \right\} \tag{30}$$

$$\frac{(N-n+i)...(N-n+1)}{i!}\frac{f^*((N-n+i)\lambda)}{((N-n+i)\lambda)^2}$$

Moreover, the average delay time W is as follows

$$W = W_q + T_S \tag{31}$$

The working probability P_w of minimum number of elements m in the system while keeping the system operates in a normal condition is given by

$$P_w = P_0 + P_1 + ... + P_{N+1-m} \tag{32}$$

4. NUMERICAL RESULTS AND DISCUSSIONS

In previous section, we assumed in our previous calculations that the service time was general service distribution. However, to alleviate the complexity of the numerical calculations, we consider that the service time is Erlang k-type distribution, where k is any arbitrary positive integer, henceforth, the equation (14) will be

$$\begin{aligned} f^*(i\lambda) &= \int_0^\infty f(x)e^{-i\lambda x}dx \\ &= \int_0^\infty \frac{(k\mu)^k}{(k-1)!}x^{k-1}e^{-k\mu x}e^{-i\lambda x}dx = \left(\frac{k\mu}{i\lambda+k\mu}\right)^k \end{aligned} \tag{33}$$

Equ. (33) is Laplace transformation of *f(x)*, where μ is any arbitrary positive service rate. We analyze the initial system performance by evaluating (11) to (13), and for example, we chose N=4 and normalize $T_s = 1/\mu$=1 with k=2, the system performance is plotted in Fig.4.

The Figure shows the state probability p_0 when all elements operate entirely, it means that there is no defective elements in the system or n=0. The state probability p_0 is sharply decreased as arrival rate increased while the state probability $p_n for(n=N+1)$ is increasing. However, each state probability pn for (0< n <N+1) has a maximum value at a specific rate of λ. These maximum points for p_n become higher as k increased.

As we mentioned above, the number of faulty elements should be less than or equal to $m(m\epsilon N+1)$ to keep the system operates normally. Fig. 5 illustrates the average system length L and the average waiting time Wq in a normal systems condition for k=2 versus arrival rate λ. The queue length L_q and the average waiting time W_q are gradually increased as arrival rate λ is slightly increased.

Fig.6 illustrates the probability of the system which operates normally, or we can say working probability P_w, versus arrival rate of the defective elements with the effect of k on the Erlang k-type distributed. However, if the system has no fault tolerance capabilities then any defective element in the system will cause a systems termination.

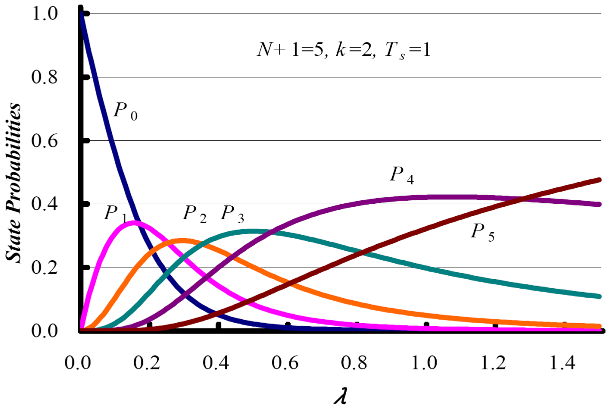

Fig.4: *The state probabilities p_n versus the arrival rate λ on condition of N+1=5, k=2, T_S=1 (unit time)*

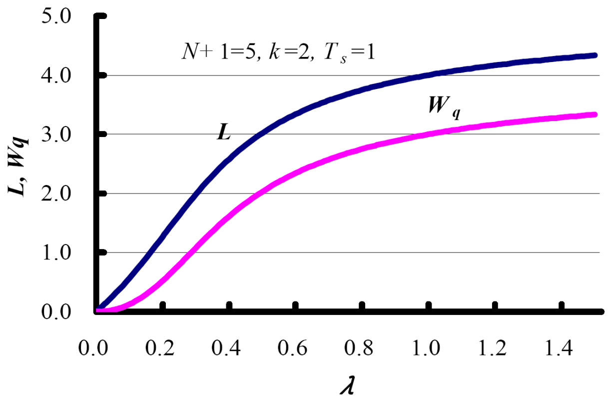

Fig.5: *The average queue length L_q and the average waiting time W_q versus the arrival rate λ on condition of N+1=5, k=2, T_S=1 (unit time)*

For example, suppose that the system has only 5 elements in operations then any element failure will cause system termination with probability p_0 =0.269 at λ=0.2 with k=1. In a condition of the system has a fault tolerance capability and suppose that the system has 3 of 5 elements are out of services then we will able to calculate the working probability Pw=0.798 at λ=0.2 with k=1. Therefore, our model has fault tolerance and has the ability to respond reasonably to an unexpected failure, and we can realize that systems operation works normally even the system has some defective elements.

To bring the Erlang k-type distribution closer to general distribution, let maneuver the multi-type of Erlang distributions (m-type) as follows.

$$f(x) = \sum_{i=1}^{m} \alpha_i \frac{k_i \mu_i}{(k_i-1)!} x^{k_i-1} e^{-k_i \mu_i x} \quad \left(\sum_{i=1}^{m} \alpha_i = 1 \right) \tag{34}$$

Equation (15) becomes as follows

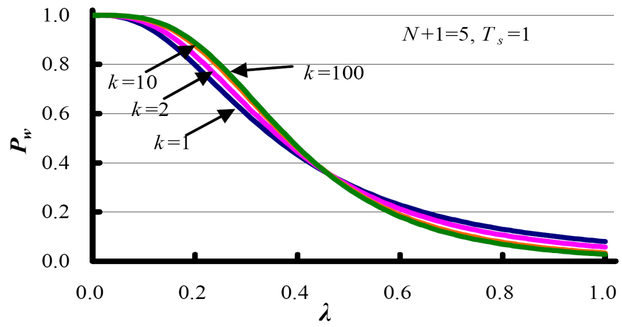

Fig.6: *The probability P_w of the 3 out of 5 system versus the arrival rate λ on conditions of N+1=5, T_S=1 (unit time).*

$$T_S = \int_0^\infty x f(x) dx = \sum_{i=1}^{m} \alpha_i \cdot \frac{1}{\mu_i} \tag{35}$$

For example and as illustrated in Fig. 7, two types of Erlang k-type distributions are convoluted for ($k_1 = 2, \mu_1 = 1 \alpha_1 = 0.4$) and ($k_2 = 10, \mu_2 = 0.2, \alpha_2 = 0.6$). The figure shows that the first type Erlang k1-type distribution is a simple failure and the time service processing is a short service period, and the second one is a complex failure and the time service processing is longer. The combination of these two processes is given by

$$f(x) = \alpha_1 \frac{k_1 \mu_1}{(k_1-1)!} x^{k_1-1} e^{-k_1 \mu_1 x} + \alpha_2 \frac{k_2 \mu_2}{(k_2-1)!} x^{k_2-1} e^{-k_2 \mu_2 x} \tag{36}$$

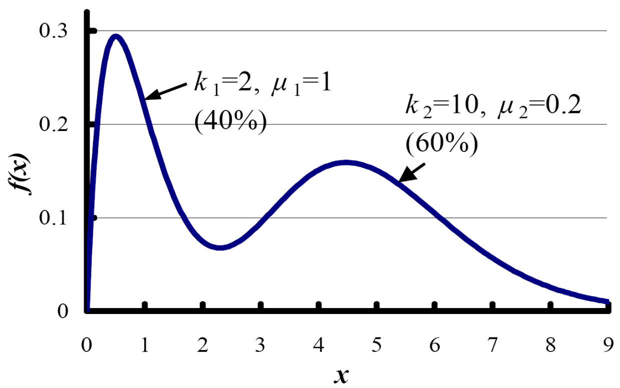

Fig.7: *Probability density function f(x).*

Using above formula to substitute *f(x)* in (14) to calculate other unknown values. The state probabilities p_n of *CFBLT* for $N+1 = 5, k_1 = 2, k_2 = 10, \mu_1 = 1, \mu_2 = 0.2, \alpha = \alpha_1 = 0.4, \alpha_2 = 1 - \alpha$can be defined. The first distribution of ($k_1 = 2, \mu_1 = 1$) has short service time (x) and can be neglected, which the system is able to operate normally.

However, the second distribution of ($k_2 = 10, \mu_2 = 0.2$) has a major impact on the average service time which is significantly long and the systems operation is required to be monitored carefully. Therefore, our analysis brings better understanding systems behavior to take a preceding step for keeping the system performs normally.

5. CONCLUSION

A study of a close systems activities is very important because of its widely used in recent computer systems and in workplaces. In this paper, we presented an analytical method for a closed feed back loop type *CFBLT* queuing model, which is appropriated for failure and repair processes in maintenance fields.

Numerical examples were given to gain a better understanding of the systems behaviors. Furthermore, the system model has a fault tolerance capability by considering the system has a self configuration feature that can tolerant temporary failures while tasks which have been assigned to a failure element are diverted and distributed to other active elements.

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