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Contributed Paper

Exact Solution for Average Run Length of CUSUM Charts for MA(1) Process

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Abstract

In this paper we apply Fredhom type integral equations method to derive explicit formula of the average run length (ARL) for a Cumulative Sum (CUSUM) chart, when observations are described by a first order moving average MA(1) process, with exponential white noise. We compare the computational time between our analytical explicit expressions for the ARL performance with the one obtained via Gauss-Legendre numerical scheme for integral equations. We found that those methods are in excellent agreement however, the computational time of the former takes approximately 1 second while the latter method consumes the computational time 11 minutes approximately.

Keywords: statistical process control, cumulative sum, moving average observation, average run length, Fredholm type integral equations, numerical approximations

1. Introduction

The Cumulative Sum (CUSUM) chart is a simple and very effective graphical procedure for monitoring the quality control in manufacturing industry. The CUSUM chart was first introduced by Page [1] to detect a change in observed parameters and widely implemented in statistical process control. Some recent reviews are given in the paper of Mazalov and Zhuravlev [2], who implemented the CUSUM chart to identify the changing point in a traffic network. CUSUM chart was intensively used by Ben et al. [3] in environmental science to detect mean changes in air pollution. Kennedy [4] applied in queuing process computed the distribution of the first passage times for a M/M/l queue and stopping times associated with sequential Cumulative Sum tests. In addition, there are many applications of CUSUM chart in health care and public health (see Lim et al. [5], Noyez [6]).

The common characteristic of any control chart is the Average Run Length (ARL), defined as the expectation of an alarm times taken to trigger a signal about a possible change in parameters distribution. Ideally, an acceptable ARL of an in-control process should be large enough to detect a small change in parameter of distribution. In this paper we adopt the following notation $ARL_0 = \mathsf{E}_\infty(\tau) = T$ where $\mathsf{E}_\infty(.)$ is the expectation corresponding to the target value and is assumed to be large enough. The ARL when the process is out-of-control is called the Average Delay time denoted by ARL_1, defined as the expectation of delay for true alarm time. This time should minimize the quantity

$$ARL_1 = \hat{\mathsf{E}}_\nu\left(\tau - \nu + 1 \mid \tau \geq \nu\right)$$

where $\hat{E}_\nu(.)$ is the expectation under the assumption that a change-point occurs at the beginning time.

In literature, there are several methods for evaluating ARL_0 and ARL_1 for the CUSUM and EWMA procedures have been studied. These methods are the Monte Carlo simulations, the Integral Equations (IE) approach (see Varderman and Ray [7], Crowder [8], Srivastava and Wu [9]) the Markov Chain Approximation (MCA) (see Brook and Evan [10]), Lucass and Saccucci [11]. Recently, Areepong [12] proposed analytical derivation to find explicit formulas for ARL of the EWMA chart when observations are exponential distributed. Mititelu et al. [13], presented analytical expressions to determine the ARL of the EWMA and CUSUM chart when observations are from hyperexponential distribution via Fredholm integral equations approach. Petcharat et al. [14], [15] derived the closed form expressions for the ARL of the CUSUM chart when observations are Pareto and Weibull distributed by approximation these distributions with a hyperexponential distribution. Traditionally, the CUSUM charts have been designed when observations are independent and identically distributed (i.i.d). However, in real life problems, correlated observations may be presented in some process see Wardell et al. [16], Yashchin [17], and Zhang [18]. Correlations may affect the properties of the CUSUM chart (see Lu and Reynold [19]). Atieza et al. [20] applied the CUSUM chart on residuals of a time series model with process observations described by a normal distribution. Jacob and Lewis [21] analyzes autoregressive–moving average process order 1,1 denoted by ARMA(1,1), when observations are exponentially distributed with exponential white noise.

The work of Lawrance and Lewis [22] presented exponential moving average of order 1. Such models are important in queuing and network process. Mohamed and Hocine [23] proposed a Bayesian analysis of the autoregressive model with exponential white noise.

In this paper, we derive analytically explicit formulas for ARL_0 and ARL_1 for CUSUM chart when observations are first order of moving average process, MA(1), with exponential white noise. Next, in section 2, we formulate the problem of existence and uniqueness of solution using Banach's fixed point (Venkateshwara Rao et al. [24]). In section 3, we show the analytical derivation of the results, and compare the computational time between the closed form expressions and the numerical approximations.

2. THE AVERAGE RUM LENGTH (ARL) INTEGRAL EQUATIONS FOR CUSUM CHART OF FIRST ORDER MOVING AVERAGE (MA(1)) PROCESS WITH EXPONENTIAL WHITE NOISE

The CUSUM chart is often implemented in monitoring and detecting small change in parameter of a distribution. Let ξ_n be a sequence of independent and identically distribution (i.i.d.) nonnegative random variables defined by the recurrence

$$X_n = \max(X_{n-1} + \xi_n - a, 0), \quad n = 1, 2, \ldots \quad (1)$$

where ξ_n are random variables and a is non-zero constant. The corresponding stopping time for the CUSUM scheme is described by Eq.(1) is defined as

$$\tau_b = \inf\{t > 0; X_n > b\} \quad (2)$$

where b is a constant parameter known as the control limit.

Let $\mathbf{P}_X$ and $\mathbf{E}_X$ be the probability measure and the induced expectation corresponding to the initial value $X_0 = x$. Then, it can be shown see [24], that $ARL = j(x) = \mathbf{E}_x \tau_b < \infty$ is the unique solution of the following Fredholm type integral equation

$$j(x) = 1 + \mathbf{E}_X\left[I\{0 < X_1 < b\} j(X_1)\right]$$

$$+\mathbf{P}_X\{X_1 = 0\} j(0), \quad b > x. \quad (3)$$

2.1 The Uniqueness of the Solution for the ARL Integral Equation

The case when are continuous distributed i.i.d. random variables, with exponential distribution was described in Mititelu et al. [13]. The case of a stationary first order autoregressive process with exponential white noise process was analyzed by Busaba et al. [25]. In this paper, we focus on a stationary first order moving average process, MA(1) with exponential white noise ξ_n defined by the recurrence $Z_n = \xi_n - \theta\xi_{n-1}$ where $-1 < \theta < 1$ and $\xi \sim Exp(\lambda)$. In this case the Eq.(3) can be written as

$$j(x) = 1 + \lambda e^{\lambda(x-a-\theta\xi_0)} \int_0^b j(y) e^{-\lambda y} dy$$

$$+\left(1 - e^{-\lambda(a-x+\theta\xi_0)}\right) j(0), x \in [0, a). \quad (4)$$

Since the right hand side of Eq.(4) is continuous such that the solutions of the integral Eq.(4) is continuous function.

Let now consider the complete metric space $(C(I), \| \|_1)$ where $(C(I), \| \|_1)$ denotes the space of all continuous function over a compact interval I with the sup norm, i.e., $\|j\| = \sup_{x \in I} |j(x)|$.

Recall that an operator T is named a contraction, if there exist a number $0 < q \le 1$ such that $\|T(j_1) - T(j_2)\| \le q\|j_1 - j_2\|$ for all $j_1, j_2 \in I$.

Let T be an operator in the class of all continuous function $C(I_k)$ where $I_k = [(k-1)a, ka]$, for any $k = 1, \ldots, n$ defined by

$$T(j(x)) = 1 + \lambda e^{\lambda(x-a-\theta\xi_0)} \int_0^b j(y) e^{\lambda y} dy$$

$$+\left(1 - e^{-\lambda(a-x+\theta\xi_0)}\right) j(0). \quad (5)$$

Then, the integral equations in Eq.(5) can be written in operatorial form as $T(j)(x) = j(x)$. According to Banach's Fixed Point Theorem, if the operator T is a contraction then the fixed point equation $T(j)(x) = j(x)$ has a unique solution. To prove uniqueness solution of Eq.(5), we will show in Theorem 2.1 that T is contraction. First, define the norm $\|j\|_1 = \sup_{x \in C(I_1)} |j(x)|$.

Theorem 2.1 On the metric space $(C(I_1), \| \|_1)$ the operator T is contraction.

Proof. First, to show that T is a contraction we need to check that for any $x \in I_1$ and $j_1, j_2 \in C(I_1)$ we have the inequality

$$\|T(j_1) - T(j_2)\|_1 \le q\|j_1 - j_2\|_1,$$

where $0 \le q < 1$. According to Eq. (5) we have that:

$$\|T(j_1) - T(j_2)\|_1$$

$$\le \sup_{x \in [0,a)} \left| j_1(0) - j_2(0)(1 - e^{-\lambda(a-x+\theta\xi_0)} + \lambda e^{-\lambda(a-x+\theta\xi_0)} \int_0^b (j_1(y) - j_2(y\right.$$

$$\le \sup_{x \in [0,a)} \|j_1 - j_2\|_1 (1 - e^{-\lambda(a-x+\theta\xi_0)})$$

$$+\|j_1 - j_2\|_1 \lambda e^{-\lambda(a-x+\theta\xi_0)} \int_0^b e^{-\lambda y} dy$$

$$= \|j_1 - j_2\|_1 \sup_{x \in [0,a)} \left[1 - e^{-\lambda(a-x+\theta\xi_0) - \lambda b}\right]$$

$$= \left[1 - e^{-\lambda\theta z_0 - \lambda b}\right] \|j_1 - j_2\|_1$$

$$= q_1 \|j_1 - j_2\|_1,$$

where $q_1 = \left[1 - e^{-\lambda\theta\xi_0 - \lambda b}\right] < 1$. □

We have used triangular inequality and the fact that

$$\left|j_1(0)-j_2(0)\right| \le \sup_{x\in[0,a)} \left|j_1(x)-j_2(x)\right| = \left\|j_1-j_2\right\|_1 .$$

Therefore, the uniqueness of solution is guaranteed by Theorem 2.1. Next, in Theorem 2.2, we will derived explicit solution of Fredholm integral from Eq.(4).

Theorem 2.2 The solution of Eq.(4) is

$$j(x) = e^{\lambda b}\left(1+e^{(a+\theta\xi_0)}-\lambda b\right)-e^{\lambda x}; x\ge 0. \quad (6)$$

Proof. $j(x) = 1+\lambda e^{\lambda(x-a-\theta\xi_0)}\int_0^b j(y)e^{-\lambda y}dy$

$$+\left(1-e^{-\lambda(a-x+\theta\xi_0)}\right)j(0),\ x\grave{o}[0,a). \quad (7)$$

Let d be constant as $d=\int_0^b j(y)e^{-\lambda y}dy.$ $j(x)$ can be written as

$$j(x)=1+\lambda e^{\lambda(x-a-\theta\xi_0)}d$$
$$+\left(1-e^{-\lambda(a-x+\theta\xi_0)}\right)j(0),\ x\grave{o}[0,a). \quad (8)$$

For $x=0$ then

$$j(0)=1+\lambda e^{\lambda(-a-\theta\xi_0)}d+\left(1-e^{-\lambda(a+\theta\xi_0)}\right)j(0)$$
$$=e^{\lambda(a+\theta\xi_0)}+\lambda d .$$

Then

$$j(x)=1+\lambda e^{\lambda(x-a-\theta\xi_0)}d$$
$$+\left(1-e^{-\lambda(a-x+\theta\xi_0)}\right)(e^{\lambda(a+\theta\xi_0)}+\lambda d), d=\int_0^b j(y)e^{-\lambda y}dy$$
$$=1+e^{\lambda(a+\theta\xi_0)}+\lambda d-e^{\lambda x}$$
$$=1+\lambda d+e^{\lambda(a+\theta\xi_0)}-e^{\lambda x}. \quad (9)$$

Now, constant d can be found as following:

$$d=\int_0^b j(y)e^{-\lambda y}dy$$
$$=\int_0^b j\left(1+\lambda d+e^{\lambda(a+\theta z_0)}-e^{\lambda y}\right)e^{-\lambda y}dy$$
$$=\left(1+\lambda d+e^{\lambda(a+\theta z_0)}\right)\int_0^b e^{-\lambda y}dy-\int_0^b e^{\lambda y-\lambda y}dy$$
$$=\frac{e^{\lambda b}}{\lambda}\left(1-e^{-\lambda b}\right)\left(1+e^{\lambda(a+\theta\xi_0)}\right)-be^{\lambda b} .$$

Substituting the constant d into Eq.(9), then

$$j(x)=e^{\lambda b}\left(1+e^{\lambda(a+\theta z_0)}-\lambda b\right)-e^{\lambda x},\quad x\ge 0. \quad (10)$$

2.2 Numerical Solution for the ARL Integral Equation

In this section, we present a numerical method to evaluate the solutions of Fredholm type integral Eq.(3). It can be shown that the ARL of the CUSUM chart. The $j(x)=\mathrm{E}_x\tau_b$, is a solution of the integral equation Eq.(4), then

$$j^{IE}(x)=1+j(0)F(a-x+\theta\xi_0),$$
$$+\int_0^b j(y)f(a-x+\theta\xi_0+y)dy \quad (11)$$

where $F(x)=1-e^{-\lambda x}$ and $f(x)=\frac{dF(x)}{dx}=\lambda e^{-\lambda x}$.

Now, we can approximate the integral $j(x)$ via the Gauss-Legendre rule as

$$j(a_i)\approx 1+j(a_1)F(a-a_i+\theta\xi_0)$$
$$+\sum_{k=1}^{m} w_k j(a_k) f\left(a_k+a-a_i+\theta\xi_0\right) \quad (12)$$

with the weights $w_k=\frac{b}{m}\ge 0$ and

$$a_k=\frac{b}{m}\left(k-\frac{1}{2}\right);\ ;k=1,2,...,m.$$

In the MA(1) process with exponential white noise, the numerical solution for ARL integral equation can be written as follows

$$j(a_i)=1+j(0)F(a-x+\theta z_0) \quad (13)$$

We approximate the numerical integration by sum of areas of rectangles with bases $\frac{b}{m}$ with heights chosen as the value of $f(a_k)$ at the midpoints of intervals of length $\frac{b}{m}$ beginning at zero. Then, on the interval with the division points and weights then, on the interval $[0,b]$ with the division points $0\le a_1\le a_2\le ...\le a_m<b$ and weights $w_k=\frac{b}{m}\ge 0$ then

$$\int_0^b j(y)dy\approx\sum_{k=1}^{m} w_k f(a_k)$$

where $a_k=\frac{b}{m}\left(k-\frac{1}{2}\right);\ k=1,2,...,m. \quad (14)$

The integral in Eq.(14) is a system of m linear equations in the m unknowns $j(a_1), j(a_2),..., j(a_m)$ can be written as

$$\begin{cases} j(a_1)=1+j\left(a_1\right)F(a-a_1+\theta\xi_0)+w_1 f(a+\theta\xi_0)+\sum_{k=2}^{m} w_k j(a_k) f\left(a_k+a-a_i+\theta\xi_0\right) \\ j(a_2)=1+j\left(a_1\right)F(a-a_2+\theta\xi_0)+w_1 f(a_1+a-a_2+\theta\xi_0)+\sum_{k=2}^{m} w_k j(a_k) f\left(a_k+a-a_i+\theta\xi_0\right) \\ \cdot \\ \cdot \\ \cdot \\ j(a_m)=1+j\left(a_1\right)F(a-a_m+\theta\xi_0)+w_1 f(a_1+a-a_m+\theta\xi_0)+\sum_{k=2}^{m} w_k j(a_k) f\left(a_k+a-a_i+\theta\xi_0\right) \end{cases} \tag{15}$$

For numerical implementation is preferable writing the linear system as Eq.(15) in matrix form as follows:

$$\mathrm{J}_{m\times1} = 1_{m\times1} + \mathrm{R}_{m\times m}\mathrm{J}_{m\times1}$$

$$(\mathrm{I}_m\text{-}\mathrm{R}_{m\times m})\mathrm{J}_{m\times1} = 1_{m\times1} \tag{16}$$

where

$$\mathrm{J}_{m\times1} = \begin{pmatrix} j(a_1) \\ j(a_2) \\ \cdot \\ \cdot \\ \cdot \\ j(a_m) \end{pmatrix}, \qquad 1_{m\times1} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix},$$

$$\mathrm{R}_{m\times m} = \begin{pmatrix} F(a-a_1+\theta\xi_0)+w_1 f(a) & w_2 f(a_2+a-a_1+\theta\xi_0) & \ldots & w_m f(a_2+a-a_1+\theta\xi_0) \\ F(a-a_1+\theta\xi_0)+w_1 f(a_1+a-a_2+\theta\xi_0) & w_2 f(a+\theta\xi_0) & \ldots & w_m f(a_m+a-a_2+\theta\xi_0) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ F(a-a_m+\theta\xi_0)+w_1 f(a_1+a-a_m+\theta\xi_0) & w_2 f(a_2+a-a_m+\theta\xi_0) & \ldots & w_m f(a+\theta\xi_0) \end{pmatrix},$$

and $\mathrm{I}_m = diag(1,1,\ldots 1)$ is the unit matrix order m. If it exists $(\mathrm{I}_m - \mathrm{R}_{m\times m})^{-1}$, then the solution of Eq.(16) is

$$\mathrm{J}_{m\times1} = (\mathrm{I}_m\text{-}\mathrm{R}_{m\times m})^{-1} 1_{m\times1}.$$

Solving the set of Eq.(15) for approximate values of $j(a_1), j(a_2), \ldots, j(a_m)$ we may approximate the function $j^{IE}(x)$ as

$$j^{IE}(x) \approx 1 + j(a_1)F(a-x+\theta\xi_0) + \sum_{k=1}^{m} w_k j(a_k) f(a_k - a - a_i + \theta\xi_0), \tag{17}$$

with $w = \dfrac{b}{m}$ and $a_k = \dfrac{b}{m}\left(k - \dfrac{1}{2}\right)$.

3. Numerical Results

In this section, we compare the ARL obtained from two approaches; explicit solution denotes as $j(x)$ and approximated solution denotes as $j^{IE}(x)$. The relative error is calculated by

$$\varepsilon_r = \frac{\left|j(x) - j^{IE}(x)\right|}{j(x)} \times 100. \tag{18}$$

Table 1 shows relative errors less than 0.2% between the proposed formula and the numerical integration by Gauss-Legendre numerical method with $m = 500$ nodes. The computational time based on our formula takes less than 1 second while the numerical integral equation takes approximately 11 minutes.

In Tables 2 and 3, the results obtained from the explicit formula and IE are shown in term of a comparison of the ARL by fixed , , and the number of division points Notice that is the in-control parameter then the first row shows the values of while the parameter correspond to the values of out-of-control parameters.

Table 1. Comparison of ARL obtained from explicit formula and numerical integral equation (IE) when $\lambda = 1$ and $m = 500$.

θ	b	ARL	$a = 3.5$		$a = 4$	
			$x = 0$	$x = 2$	$x = 0$	$x = 2$
0.23	0.38	Explicit	60.853	54.464	100.391	94.002
		IE	60.381	54.4	100.353	93.967
		Time :min	11.09	11.08	11.09	11.07
		ε_r	0.776	0.118	0.038	0.037

Table 1. Continued.

θ	b	ARL	$a=3.5$		$a=4$	
			$x=0$	$x=2$	$x=0$	$x=2$
0.23	1.7	Explicit	223.317	216.928	371.323	364.943
		IE	222.947	216.569	370.701	364.322
		Time :min	11.13	11.13	11.04	11.05
		ε_r	0.166	0.165	0.168	0.170
	2.0	Explicit	299.580	293.191	499.366	492.977
		IE	298.995	292.619	498.381	492.005
		Time :min	11.16	11.20	11.09	11.09
		ε_r	0.195	0.195	0.197	0.197
0.53	0.38	Explicit	82.1761	75.787	135.546	129.157
		IE	82.1454	75.759	135.495	129.108
		Time :min	11.34	11.31	11.14	11.25
		ε_r	0.037	0.037	0.038	0.038
	1.7	$j(x)$	303.138	296.7448	502.924	496.535
		IE	302.631	296.253	502.078	495.699
		Time :min	11.28	11.28	11.29	11.29
		ε_r	0.167	0.166	0.168	0.168
	2.0	Explicit	407.326	400.937	677.009	670.620
		IE	406.525	400.149	675.668	669.292
		Time :min	11.3	11.18	11.30	11.29
		ε_r	0.197	0.197	0.198	0.198
0.83	0.38	Explicit	110.959	104.570	183.001	176.612
		IE	110.917	104.531	182.932	176.545
		Time :min	11.12	11.35	10.88	11.12
		ε_r	0.038	0.037	0.038	0.038
	1.7	Explicit	410.883	404.494	680.566	674.177
		IE	410.194	403.816	679.418	673.04
		Time :min	11.40	11.45	11.16	11.17
		ε_r	0.168	0.168	0.169	0.169
	2.0	Explicit	552.768	546.278	916.802	910.413
		IE	551.676	545.299	914.981	908.605
		Time :min	11.37	11.36	11.23	11.11
		ε_r	0.198	0.179	0.199	0.199

Table 2. Comparison of ARL obtained from explicit formula and numerical integral equation (IE) when $a=4$, $b=1.7$ and $m=500$.

λ	$\theta=0.23$		ε_r
	Explicit	IE	
1.0	371.323	370.701	0.168
1.1	215.845	215.518	0.151
1.2	137.285	137.097	0.137
1.3	93.5929	93.4754	0.126
1.4	67.3893	67.3116	0.115
1.5	50.6946	50.6407	0.106

Table 3. Comparison of ARL obtained from explicit formula and numerical integral equation (IE) when $a=4$, $b=2$ and $m=500$.

λ	$\theta=0.23$		ε_r
	Explicit	IE	
1.0	499.366	498.381	0.197
1.1	281.652	282.154	0.178
1.2	174.955	175.238	0.162
1.3	116.898	117.071	0.148
1.4	82.7262	82.8386	0.136
1.5	61.3045	61.3812	0.125

4. Conclusions

We present the explicit formula to evaluate the ARL_0 and ARL_1 of the CUSUM chart when observations are the first order moving average process, MA(1) with exponential white noise distribution. The accuracy and computational speed for analytical formula was compared with values obtained from numerical integration method by Fredholm integral equation. We have shown that our analytical expressions are very accuracy in the same manner of the numerical integration methods, however, in terms of the computational time the latter method is very time consuming.

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