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Contributed Paper

Local Cohomology and Sequentially Generalized Cohen-Macaulay Modules

Amir Mafi

Department of Mathematics, University of Kurdistan, Pasdaran St., P.O. Box: 416, Sanandaj, Iran.

*Author for correspondence; e-mail: a_ mafi@ipm.ir

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ABSTRACT

Let (R, m) be a commutative Noetherian local ring and M a finitely generated R-module with dim M = d. It is shown that M is a sequentially generalized Cohen-Macaulay module if and only if the local cohomology modules $H_m^j(M)$ are either of finite length or generalized co-Cohen-Macaulay of Noetherian dimension j for all $0 \leq j \leq d-1$.

Key words: sequentially generalized cohen-macaulay modules, local cohomology modules

1. INTRODUCTION

Throughout this paper, we assume that (R, m) is a commutative Noetherian local ring and M a non-zero finitely generated R-module with dim M = d. The notion of generalized Cohen-Macaulay modules introduced by Cuong, Schenzel and Trung [6] is the first extension of Cohen-Macaulay modules. The theory of generalized Cohen- Macaulay modules was developed rapidly in the 1980s and early 1990s by the works of many authors and found its applications in many fields of commutative algebra and algebraic geometry. A further generalization of Cohen-Macaulay modules is the notion of sequentially Cohen-Macaulay modules introduced first by Stanley [13] for graded modules (see also [7]). After that, this notion was defined for modules over local rings by Schenzel [12], Cuong and Nhan [5]. The module M is called a sequentially Cohen-Macaulay module if there exists a filtration F: $0=M\_0 \subset M_1 \subset \ldots$ $M_t = M$ of submodules of M such that each quotient $M_i/M_i{-}1$ is Cohen-Macaulay and $\dim M_1/M_0 < \dim M_2/M_1 < \ldots < \dim M_t / M_t{-}1$. Cuong and Nhan [5] also introduced the notion of sequentially generalized Cohen-Macaulay modules and gave a characterization for these modules. The definition of sequentially generalized Cohen- Macaulay module is similar to the one of sequentially Cohen-Macaulay module except each module $M_i/M_i{-}1$ is required to be a generalized Cohen-Macaulay module instead of being Cohen-Macaulay. Then, the filtration F of the sequentially generalized Cohen-Macaulay module M is called the generalized Cohen-Macaulay filtration. The filtration F of M is said to be the dimension filtration if $M_i{-}1$ is the largest submodule of M_i which has dimension strictly less that $\dim M_i$ for i=1, 2,..., t. A filtration of sequentially generalized Cohen-Macaulay is determined uniquely up to m-primary components, i.e. if $0=M_0 \subset M_i \subset \ldots \subset M_s = M$ is the dimension filtration of M and $0 = N_0 \subset N_1 \subset \ldots \subset N_t = M$ is a generalized Cohen-Macaulay filtration then t=s and $l(M_i)/N_i < \infty$ for all i=1, 2,..., t-1. Therefore in this case the dimension filtration is also a generalized Cohen-Macaulay filtration.

The notion of local homology modules was defined in [3] as follows: Let a be an ideal of R and N an arbitrary R-module. The ith local homology module $H_i^a(N)$ of N with respect to a is defined by

$$H_i^a(N) = \lim_{\bar{n}} Tor_i^R(R/_{a^n}, N).$$

Many basic properties of local homology modules for Artinian modules were presented in [3] and [4], which show that this theory of local homology modules is in some sense dual to the theory of local cohomology for finitely generated modules. Cuong, Dung and Nhan [2] defined that an Artinian R-module A is generalized co-Cohen-Macaulay if the ith local homology module $H_i^a(N)$ of A is of finite length for all i < N-dim A, where N-dim A is the Noetherian dimension of the Artinian module A defined by Roberts [11] and Kirby [8]. It is clear that the class of generalized co-Cohen-Macaulay modules contain strictly all co-Cohen-Macaulay modules defined in [14].

The main aim of this paper is to prove the following theorem.

Theorem 1.1 Let $0= M_0 \subset M_i \subset \ldots \subset M_t = M$ be the dimension filtration of M and $d_i = \dim M_i/M_i-1$ for i = 1,2,..., t. Then the following statements are equivalent:

(a) M is sequentially generalized Cohen-Macaulay;
(b) for all j =0, 1,..., d, the local cohomology modules $H_m^j(M)$ are either of finite length or generalized co-Cohen-Macaulay of Noetherian dimension j;
(c) for all j =0, 1,..., d − 1 the local cohomology modules $H_m^j(M)$ are either of finite length or generalized co-Cohen-Macaulay of Noetherian dimension j.

It is clear that this theorem extends [9, Theorem 3.7].

2. Results

The following lemma follows immediately from the definition.

Lemma 2.1

(a) Suppose that M is sequentially generalized Cohen-Macaulay with filtration $0=M_0 \subset M_1 \ldots M_t=M$. Then the module M/Mi is sequentially generalized Cohen-Macaulay with filtration $0=M_i/M_i\ \ M_i+1\ /M_i \ldots M_t/M_i$ for any i =0, 1,..., t.
(b) Suppose that $M_1 \subset M$ and M_1 is generalized Cohen-Macaulay and M/M_1 is sequentially generalized Cohen-Macaulay with $\dim M_1 < \dim M/M_1$. Then M is sequentially generalized Cohen-Macaulay.

Lemma 2.2 Suppose that M is sequentially generalized Cohen-Macaulay with filtration $0=M_0 \subset M_1 \subset \ldots \subset M_t=M$ and assume that $d_i = \dim M_i/M_i-1$ for i= 1,2, ... , t. Then $l(H_m^j(M)) < \infty$ for all $j \notin \{d_1,\ldots, d_t\}$.

Proof. We prove by induction on t. If t = 1, then by [1, Exercise 9.5.7] $l(H_m^j(M)) < \infty$ for all $j \neq d_1$ and the result true in this case. Let t > 1. By [1, Exercise 9.5.7] we have $l(H_m^j(M)) < \infty$ for all $j \neq d_1$. Therefore from the exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

we get the exact sequence

$$\ldots\to H_m^{d_1-1}(M_1) \to H_m^{d_1-1}(M) \to H_m^{d_1-1}\left({}^{M}/_{M_1}\right) \to H_m^{d_1}(M_1) \to H_m^{d_1}(M) \to H_m^{d_1}({}^{M}/_{M_1}) \to 0 \ (\dagger)$$

and isomorphisms $H_m^j(M)/M_1) \cong H_m^j(M)$ for all $j > d_1$. By Lemma 2.1, the module M/M_1 is sequentially generalized Cohen-Macaulay and has a generalized Cohen-Macaulay filtration of length $t-1$. Hence by the induction hypothesis we have $l(H_m^j(M)/M_1)) < \infty$ for all $j \notin \{d_2,\ldots,d_t\}$. This implies that $l(H_m^j(M)/M_1)) < \infty$ for all $j < d_1$. Hence by the exact sequence (†) in conjunction with $l(H_m^j(M_1)) < \infty$ for all $j \neq d_1$ we have $l(H_m^j(M)) < \infty$ for all $j \notin \{d_1,\ldots,d_t\}$, as required. ■

Following Roberts [11] introduced the concept of Krull dimension (Kdim) for Artinian modules. Later Kirby [8] changed the terminology of Roberts and referred to Noetherian dimension (N-dim) to avoid any confusion. Here we use the terminology of Kirby [8]. The Noetherian dimension of an Artinian module A, denoted by N-dim(A), is defined inductively as follows. When A = 0, put N-dim(A) = −1.

Lemma 2.3 Let $P \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} Q$ an exact sequence of Artinian modules such that P and Q are of finite length. Then A is generalized co-Cohen-Macaulay if and only if B is generalized co-Cohen-Macaulay.

Proof. Let N-dim A = N-dim B = n. We get, from the above exact sequence, two short exact sequences

$0 \to \text{im } f \to A \to \text{im } g \to 0$

$0 \to \text{im } g \to B \to \text{im } h \to 0$.

Since $l(\text{im } f) < \infty$ and $l(\text{im } h) < \infty$. Therefore, by [3, Corollary 4.2] and [4, Remark 3.1(ii)], it follows that $l(H_i^a(A)) < \infty$ for all $i < n$ if and only if $l(H_i^a(B)) < \infty$ for all $i < n$. Hence A is generalized co-Cohen-Macaualy if and only if B is generalized co-Cohen-Macaualy. ■

The following lemma follows from [2, Corollary 4.9(ii)] and[10, Theorem 3.13].

Lemma 2.4 Let M be a generalized Cohen-Macaulay module of dimension d. Then $H_m^d(M)$ is generalized co-Cohen-Macaulay of Noetherian dimension d.

Theorem 2.5 Suppose that M is sequentially generalized Cohen-Macaulay with filtration $0 = M_0 \subset M_1 \ldots M_t = M$ and assume that $d_i = \dim M_i/M_i - 1$ for i = 1, 2,..., t. Then $H_m^{d_i}(M)$ is generalized co-Cohen-Macaulay of Noetherian dimension di for all i = 1, 2,..., d.

Proof. From the exact sequence

$0 \to M_{t-1} \to M \to M/M_{t-1} \to 0$

we get the exact sequence

$$H_m^{j-1}\left({}^{M}/_{M_{t-1}}\right) \to H_m^j(M_{t-1}) \to H_m^j(M) \to H_m^j({}^{M}/_{M_{t-1}}) \ (\ddagger)$$

and $H_m^d(M) \cong H_m^d({}^{M}/_{M_{t-1}})$. By using Lemma 2.4 $H_m^d(M)$ is generalized co-Cohen- Macaulay of Noetherian dimension d and by using Lemma 2.3 and the exact sequence (‡)

$H_m^j(M)$ is generalized co-Cohen-Macaulay of Noetherian dimension j if and only if $H_m^j(M_{t-1})$ is generalized co-Cohen-Macaulay of Noetherian dimension j. Similarly, by apply to the exact sequence $0 \to M_{t-2} \to M_{t-1} \to M_{t-1}/M_{t-2} \to 0$ with notice that M_{t-1}/M_{t-2} is generalized Cohen-Macaulay, we have the exact sequence

$$H_m^{j-1}(M_{t-1}/M_{t-2}) \to H_m^j(M_{t-2}) \to H_m^j(M_{t-1}) \to H_m^j(M_{t-1}/M_{t-2})$$

(£)

and

$H_m^{d_{t-1}}(M_{t-1}) \cong H_m^{d_{t-1}}(M_{t-1}/M_{t-2})$

Hence $H_m^{d_{t-1}}(M_{t-1})$ is generalized co-Cohen-Macaulay of Noetherian dimension d_{t-1} and so $H_m^{d_{t-1}}(M)$ is generalized co-Cohen-Macaulay of Noetherian dimension d_{t-1}. On the other hand by the exact sequence (£) and by Lemma 2.3 $H_m^j(M_{t-2})$ is generalized co-Cohen-Macaulay of Noetherian dimension j if and only if $H_m^j(M_{t-1})$ is generalized co-Cohen-Macaulay of Noetherian dimension j. Hence by the above arguments $H_m^j(M_{t-2})$ is generalized co-Cohen-Macaulay of Noetherian dimension j if and only if $H_m^j(M)$ is generalized co-Cohen-Macaulay of Noetherian dimension j. Continuing this process, we get the result. ∎

In [5] Cuong and Nhan introduced the notion of pseudo generalized Cohen-Macaulay modules (cf. [5, Definition 2.2]). Let N denote the largest submodule of M with dimension less than dim M. Then (cf. [5, Theorem 3.1]), M/N is a generalized Cohen-Macaulay module.

We now prove **Theorem 1.1.**

Proof of Theorem 1.1 (a)⇒(b) follows from Lemma 2.2 and Theorem 2.5.
(b)⇒(c) is obvious.
(c)⇒(a). We prove by induction on d that M is sequentially generalized Cohen-Macaulay. If d = 1, then by [12, Example (f)] there is nothing to prove. Let d > 1. Then M/M_{t-1} is generalized Cohen-Macaulay and from the exact sequence

$$0 \to M_{t-1} \to M \to M/M_{t-1} \to 0$$

we get the exact sequence

$$H_m^{j-1}(M/M_{t-1}) \to H_m^j(M_{t-1}) \xrightarrow{f^j} H_m^j(M) \xrightarrow{h^j} H_m^j(M/M_{t-1})$$

for all j = 1, 2,..., d − 1. Set X^j = ker f^j and Y^j = im h^j. Then X^j and Y^j are of finite length. Hence from the exact sequence

$$0 \to X^j \to H_m^j(M_{t-1}) \to H_m^j(M) \to Y^j \to 0$$

for all j = 1, 2,..., d−1 and using Lemma 2.3 we can conclude that M_{t-1} satisfies the hypothesis of (c). Therefore by using the induction assumption to M_{t-1}, the module M_i/M_{i-1} is generalized Cohen-Macaulay for all i = 1, 2,..., t−1. Thus M is sequentially generalized Cohen-Macaulay. This completes the proof. ∎

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