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Contributed Paper

Minimal Changes in the Right Hand Side Data for Correcting Linear Infeasibility Arising in Intensity-Modulated Radiation Therapy with the Generalized Newton Method

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Abstract

Intensity modulated radiation therapy (IMRT) gives rise to systems of linear inequalities, representing the effects of radiation on the irradiated body. These systems are often infeasible. In this paper, we perform the minimal correction using the l_2 norm by changing just the right hand vector for inconsistent set of linear inequalities which involves no negativity of the variables. We present an efficient second order algorithm using the generalized Newton method to solve this problem. A clinical case of IMRT treatment of prostate cancer is used to illustrate the solution process.

Keywords: convex optimization, generalized Newton method, intensity modulated radiation therapy, linear inequalities

1. Introduction

The fully-discretized feasibility model of the inverse problem of intensity modulated radiation therapy (IMRT) gives rise to a system of linear inequalities that describes the effects of radiation on the irradiated body and the treatment prescription, see Censor, Altschuler and Powlis [1, 2], see also Censor [3]. As an illustration, consider a simple representative system

$$A_1x \leq u^1, \quad (1)$$

$$A_2x \geq l^2, \quad (2)$$

$$x \geq 0.$$

Then we have following equivalent system:

$$Ax \leq b,$$

$$x \geq 0,$$

Where $A = \begin{bmatrix} A_1 \\ -A_2 \end{bmatrix}, b = \begin{bmatrix} u^1 \\ -l^2 \end{bmatrix}.$

In this paper we consider the following set of linear inequalities that are inconsistent:

$Ax \leq b,\ x \geq 0,$ (3)

Where $A \in R^{m\times n}$ and $b \in R^m$.

The inconsistency in system might be due to various reasons, such as lack of interaction between different groups who are defining the constraints, wrong or inaccurate estimates, error in data, over optimistic goals, and many others. Correcting system to a feasible system by minimal changes in its data has been attempted for some time. Up until now several algorithms have been developed [4, 5, 6, 7, and 8]. In the next section we present a new minimal correction method using the l_2 norm. An equivalent formulation of the problem is given and two efficient algorithms are designed to solve the new formulation. A clinical example of an IMRT treatment of prostate cancer is given to illustrate the application of the new minimal correction method.

In this work all vectors will be column vectors and we denote the *n*-dimensional real space by R^n

We mean A^T, and $||.||$, the transpose of matrix A and Euclidean norm respectively. By $(a)_+$ we mean a vector that we obtain from a by replacing the negative component by zero.

2. Norm Corrections

The minimal correction using the l_2 norm by changing the right hand side vector is:

$$\begin{array}{ll} \min\limits_{x\geq 0, r} & \frac{1}{2}\|r\|^2 \\ s.t. & Ax \leq b + r. \end{array} \quad (4)$$

In the following theorem we show that how we compute optimal *x* and *r* values.

Theorem: *Let* x^* *and* r^* *be the optimal solution of* (3). *Then* $r^* = (Ax^* - b)_+$, *and* x^* *is and optimal solution of*

$$\min_{x\geq 0}\frac{1}{2}\|(Ax-b)_+\|^2. \quad (5)$$

Proof: Let us write (3) as:

$$\begin{array}{ll} \min\limits_{x\geq 0}\min\limits_{r} \frac{1}{2}\|r\|^2 & \\ s.t. \quad Ax \leq b + r. \end{array} \quad (6)$$

Now for a given $x \in R^n$, let us first consider the inner minimization problem i.e.,

$$\begin{array}{ll} \min\limits_{r} & \frac{1}{2}\|r\|^2 \\ s.t. & Ax \leq b + r. \end{array} \quad (7)$$

It is obvious that problem (6) is a convex minimization problem. The Lagrangian of the problem (6) is given by

$$L(r,\lambda) = \frac{1}{2}\| r \|^2 - \lambda^T(Ax-(b+r)),\ \ \lambda \geq 0.$$

The KKT conditions are necessary and sufficient for optimality that are given by:

$r - \lambda = 0,$ (8)

$Ax < b + r,$ (9)

$\lambda^T(Ax - b - r) = 0,$ (10)

$\lambda > 0,$

where the vector l denotes the Lagrange multipliers. From the equation one has $r = \lambda$. From the equations (8)-(9), we have that $r^T(Ax - b - r) = 0$, $r \geq 0$. Therefore we obtain $r = (Ax - b)_+$ (see [9]). By combining these expressions, we find that the problem (3) can then be written as

$$\min_{x\geq 0}\frac{1}{2}\|(Ax-b)_+\|^2.$$

This completes the proof.

To solve (4) we use conjugate gradient algorithm and the so called generalized Newton algorithm that is discussed in the sequel.

GENERALIZED NEWTON ALGORITHM

To solve (4) we can use the logarithmic barrier [10, 11] approach by bringing the $x \geq 0$ to the objective functions as:

$$\min_x \frac{1}{2}\left\|(Ax-b)_+\right\|^2 - \mu\sum_{i=1}^{n}\log(x_i), \qquad (10)$$

where μ is the barrier parameter. We apply this method by starting from a strictly positive vector x. The logarithmic term does not allow the components of variable x to get negative.

Another approach which one might consider to solve (4) is the penalty function method as:

$$\min_x \frac{1}{2}\left\|(Ax-b)_+\right\|^2 + \frac{1}{2}c\left\|(-x)_+\right\|^2, \qquad (11)$$

where c is a big number. This does not allow to have big $\left\|(-x)_+\right\|^2$. It is worth mentioning that vector x might have very small negative values in the optimal solution which can be rounded to zero.

The objective function of (11) is piecewise quadratic, convex, and differentiable, but it is not twice differentiable.

Suppose $x, y \in R^n$, then for gradient of $f(x) = \frac{1}{2}\| (Ax-b)_+ \|^2 + \frac{1}{2}c\left\|(-x)_+\right\|^2$, we have $\left\|\nabla f(x) - \nabla f(y)\right\| \leq (\left\|A^T\right\|\left\|A\right\| + c)\left\|x-y\right\|$.

This means ∇f is globally Lipschitz continuous with constant $K = (\left\|A^T\right\|\left\|A\right\| + c)$. Thus, for this function generalized Hessian exists and is defined by the symmetric positive semidefinite matrix (see [12, 13, 14, and 9]).

However, the generalized Hessian is defined for this function that follows:

$$\partial^2 f(x) = A^T DA + c\hat{D},$$

where D and $\hat{D}$ are diagonal matrices for which $D(i,i) = 1$ when $(Ax - b)_i > 0$, $D(i,i) = 0$, when $(Ax - b)_i < 0$, and $D(i,i)$ in $[0, 1]$ when $(Ax - b)_i = 0$.

Also $\hat{D}(i,i) = 1$ when $(-x)_i > 0$, $\hat{D}(i,i) = 0$ when $(-x)_i < 0$ and $\hat{D}(i,i)$ in $[0,1]$ when $(-x)_i = 0$. Obviously the generalized Hessian is a set and for simplicity in this article we consider a specific element of this set, namely $D(i,i) = 0$ when and $\hat{D}(i,i) = 0$ when $(-x)_i = 0$. Now the generalized Newton algorithm can be outlined as follows:

Generalized Newton Algorithm

Input :
An accuracy parameter e > 0;
A starting point $x_0 \in R^n$.
Begin
$i = 0$;
While $||\nabla f(x_i)||_\infty \geq \in$ **do**
$x_{i+1} = x_i - (\partial^2 f(x_i))^{-1}\nabla f(x_i)$.
$i=i+1$;
End

Remark *In this algorithm, the generalized Hessian may be singular, thus we use a modified Newton direction Cholesky factorizations as the following:* $M^TM = (\partial^2 f(p_k)) + \gamma I_m)$, $d_k = -(M^TM)^{-1}\nabla f(p_k)$, *Where M is an upper triangular matrix,* γ *is a small positive number and* I_m *is the identity matrix of order* $_m$.

It is worth to note that one may use line search techniques such as Armijo or Wolf in the structure of the algorithm. Moreover the finite global convergence of generalized Newton algorithm with Armijo line search is proved in [15].

3. A CLINICAL EXAMPLE

Next we present a clinical IMRT example. Since prostate cancer is one of the most prevalent cancer types, we picked a prostate cancer case to illustrate the methodology.

The example we use in this study is the same as that applied for another independent method to illustrate models to handle infeasibility [16]. Here is a brief summary of the problem. The geometrical center of the prostate PTV (planning target volume) was chosen as the center of the IMRT beams. The beam angles selected for the inverse planning systems are 0°, 55°, 90°, 145°, 180°, 215°, 270° and 305°. The aperture-based inverse planning (ABIP) [17] method was applied. The aperture definition was carried out with the commercial CMS FOCUS treatment planning system [18]. For the prostate cases, a 5mm margin surrounded the clinical target volume (CTV) to define the PTV. An additional 8mm margin was added to accommodate the beam penumbra.

The apertures were selected according to the methodology of [17] and they include: fields that conform to the combined outline of all targets projected back to the radiation point source for all orientations of the treatment unit; fields that conform to the projection of the boost volume for all orientations; field segments that conform to the target but fully shields the critical structures; extra segments to adjust for the dose inhomogeneity that results from shielding critical structures that do not run along the whole length of the target. The number of apertures depended on the geometry and the topology of a particular site as well as on the complexity of the prescription, e.g., the number of boost regions. For treatment plans of prostate cancer, with bladder and rectum as the critical organs that have to be avoided, the total number of segments is usually in the range of 50-60. In this particular problem, the total number of segments is 54. The dose objectives for the prostate case are given in the next table. The lower bound is the minimum dose that has to be deposited in the organ and the upper bound is the maximum dose that the organ can tolerate. For the target volume, upper bounds on the dose are also imposed to achieve acceptable dose homogeneity. The dose values in the table 1 are in cGy.

Table 1.

	Lower Bound	Upper Bound
PTV	7800	9000
CTV	8100	9000
RECTUM	0	8000
BLADDER	0	8000

We understand that plans that meet these objectives may not be clinically acceptable. However, plans that fail these criteria definitely are not to be used. Therefore, it will be a perfect example to illustrate methods that solve infeasibility problems. The problem was initially infeasible. After application of the method described in our study, we are able to relax the bounds to a minimum and make this problem feasible. We need only to relax the bound for 69 voxels from a total of 4276 voxels, with maximum relaxation of 49.5 cGy from around 8000 cGy (< %1).

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