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Contributed Paper

On the Graded Primary Avoidance Theorem

Shahabaddin E. Atani* [a], and Unsal Tekir [b]
[a] Department of Mathematics, University of Guilan, P. O. Box 1914, Rasht. Iran.
[b] Department of Mathematics, Marmara University, Gotepe, Istanbul, Turkey.
*Author for correspondence, e-mail : ebrahimi@guilan.ac.ir

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ABSTRACT

Let G be a monoid with identity e, and let R be a G-graded commutative ring. Here we study the graded primary sub-modules of a graded R-module. While the bulk of this work is devoted to investigate the graded primary avoidance theorem for modules.
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1. INTRODUCTION

The prime avoidance theorem for modules has been introduced and studied by Chin-Pi Lu in [2]. Here we study the graded primary avoidance theorem for modules. A number of results concerning the graded primary avoidance theorem are given.

Before we state some results let us introduce some notation and terminology. Let G be an arbitrary monoid with identity. A commutative ring R with non-zero identity is G-graded if it has a direct sum decomposition (as an additive group) $R=\oplus_{g\in G} R_g$ such that $1\in R_e$ and for all $g,h\in G$, $R_g R_h \subseteq R_{gh}$. Let I be an ideal of R. For $g\in G$, let $I_g = I\cap R_g$. Then I is a graded ideal of R if $I=\oplus_{g\in G} I_g$ In this case I_g is called the g-component of I for $g\in G$.

If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M=\oplus_{g\in G} M_g$ such that for all $g,h\in G$, $R_g M_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be *homogeneous* element. A submodule $N\subseteq M$, where M is G-graded, is called G-graded R-submodule of M if $N=\oplus_{g\in G}(N\cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$. for $g\in G$. Clearly, 0 is a graded submodule of M. Also, we write $h(R)=\cup_{g\in G} R_g$ and $h(M)=\cup_{g\in G} M_g$. A graded ideal I of R is said to be *graded prime ideal* if $I\neq R$; and whenever $ab\in I$, we have $a\in I$ or $b\in I$, where $a,b\in R$. The *graded radical* of I, denoted by $Gr(I)$, is the set of all $x\in R$ such that for each $g\in G$ there exists $n_g>0$ with $x_g^{n_g}\in I$. A graded ideal I of R is said to be *graded primary ideal* if $I\neq R$; and whenever $a,b\in h(R)$ with $ab\in I$, then $a\in I$ or $b\in Gr(I)$. In this case, $Gr(I)=P$ is a graded prime ideal of R, and we say that I is a graded P-primary ideal of R. (see [7, Lemma 1.8]).

2. GRADED PRIMARY SUB-MODULES

The graded primary and primary submodules are different concepts (see [7, Example 1.6]). So we recall from [3, 4] the definitions of such submodules. Let *R* be a *G*-graded ring, *M* a graded *R*-module, *N* a graded *R*-submodule of *M*. *N* is a graded prime submodule of *M* if $N \neq M$; and whenever $a \in h(R)$ and $m \in h(M)$ with $a\,m \in N$, then either $m \in N$ or $a \in (N :_R M)$. *N* Is a graded primary submodule of *M* if $N \neq M$; and whenever $a \in h(R)$ and $m \in h(M)$ with $a\,m \in N$, then either $m \in N$ or $a^k \in (N :_R M)$ for some *K*.

The following lemma is known, but we write it here for the sake of references.

Lemma 2.1 Let I, J be graded ideals of a *G*-graded ring *R*, *M* a graded *R*-module and *N*, *K* graded *R*-submodules of *M*. The following hold:

(i) (N:M, IJ, I+J and $I \bigcap J$ are graded ideals of *R*.

(ii) If $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded submodules of *M*.

Lemma 2.2 Let R be a G-graded ring and N, K, L graded R-submodules of a graded *R*-module M with $N \subseteq K \bigcup L$. Then $N \subseteq K$ or $N \subseteq L$.

Proof. Suppose $N \subseteq K \Upsilon L$ but $N \not\subset K$ and $N \not\subset L$ Then there are homogenous elements $x_g \in N - K$ and $y_h \in N - L$ so $t = x_g + y_h \in N$; hence either $t \in K$ or $t \in L$ which is a contradiction (since K, L are graded). Thus $N \subseteq K$, or $N \subseteq L$.

Assume that *R* is a *G*-graded ring and let $N, N_1, ..., N_n$ be graded submodules of a graded *R*-module *M*. We call a covering $N \subseteq N_1 \bigcup N_2 \bigcup \cdots \bigcup N_n$ efficient if no N_k is superfluous. Analogously, we shall say that $N = N_1 \bigcup N_2 \bigcup \cdots \bigcup N_n$ is an *efficient union* if none of the N_k may be excluded. Any cover or union consisting of submodules of *M* can be reduced to an efficient one, called an *efficient reduction*, by deleting any unnecessary terms. A covering of a graded submodule by two graded submodules of a graded module is never efficient by Lemma 2.2. Thus, $N \subseteq N_1 \bigcup N_2 \bigcup \cdots \bigcup N_n$ may possibly be an efficient covering only when $n > 2$ or $n = 1$ [5].

Lemma 2.3. Assume that *R* is a *G*-graded ring and let $N = N_1 \bigcup N_2 \bigcup \cdots \bigcup N_n$ be an efficient union of graded submodules of a graded *R*-module *M* for $n > 1$. Then $\bigcap_{j \neq k} N_j = \bigcap_{j=1} N_j$ for all *K*.

Proof. It suffices to show that, say $k = 1$, $\bigcap_{j=2} N_j \subseteq \bigcap_{j=1} N_j$. Let $m = m_{g1} + ... + m_{gs} \in \bigcap_{j=2} N_j$ with $m_{gi} \neq 0$ and let n_h be a homogenous element of $N - \cup_{j=2} N_j$, so $n_h \in N_1$; hence $N_2, ..., N_n$ graded gives $m + n_h \in N - \cup_{j=2} N_j$. It follows that $m \in N_1$ since N_1 is graded, as needed.

Lemma 2.4. Let $I, I_1, ..., I_k$ be graded ideals of a *G*-graded ring *R* with *I* graded primary such that $I_i \not\subset Gr(I)$. Then $I_1 ... I_k \not\subset Gr(I)$.

Proof. Suppose not. By assumption, there are homogenous elements $x_{g_i} \in I_i - Gr(I)\,(i = 1, ..., k)$ such that $x_{g_1} x_{g_2} ... x_{g_k} \in Gr(I)$, so $Gr(I)$ graded prime (see [4, Lemma 1.8]) gives $x_{g_j} \in Gr(I)$ for some *j* which is a contradiction, as required.

Proposition 2.5. Assume that *R* is a *G*-graded ring and let $N \subseteq N_1 \cup ... \cup N_n$ be an efficient covering consisting of graded submodules of a graded *R*-module *M*. If $Gr(N_j : M) \not\subset Gr(N_k : M)$ for every $j \neq k$, then no N_k is a graded primary submodule of *M*.

Proof. Since $N \subseteq N_1 \cup ... \cup N_n$ is an efficient covering, $N = (N \cap N_1) \cup ... \cup (N \cap N_n)$ is an efficient union. Hence, there exists a homogeneous element $e_k \in N - N_k$ for every $j \in \{1,2,...,n\}$. Moreover, $\cap_{J \neq k} (N \cap N_k) \subseteq N \cap N_k$ by Lemma 2.4. If $j \neq k$, then $Gr(N_j : M) \not\subset Gr(N_k : M)$ so that there exists a homogeneous element $x_{g_i} \in Gr(N_j : M)$, but $x_{g_i} \notin Gr(N_k : M)$. Now, suppose that N_k is a graded primary submodule. Then $(N_k : M)$ is a graded primary ideal by [1, Proposition 2.7], therefore, the homogeneous element $x_h = \prod_{j \neq k} x_{g_j} \in Gr(N_j : M)$, but $x_h = Gr(N_k : M)$ by Lemma 2.5. Then there exist positive integers $s_1,...,s_n$ with $x_{g_1}^{s_1} \in (N_1 : M)$, $x_{g_2}^{s_2} \in (N_2 : M)$, ... , $x_{g_n}^{s_n} \in (N_n : M)$. Let $s = \max\{s_1,....,s_n\}$. Then $x_h^s \in (N_j : M)$ for every $j \neq k$ but $x_h^s \notin (N_k : M)$. Hence, $x_h^s e_k \in N \cap N_j$ for every $j \neq k$, but $x_h^s e_k \notin N \cap N_k$, (if $x_h^s e_k \in N \cap N_k$ then N_k graded primary gives $x_h \in (N_k : M)$; hence this is a contradiction to $\cap_{J \neq k} (N \cap N_k) \subseteq N \cap N_k$, and the proof is compele.

Theorem 2.6. [The Graded Primary Avoidance Theorem] Let *M* be a graded moddule over a *G*-graded ring *R*, $N_1,..., N_n$ a finite number of graded submodules of *M* and *N* a graded submodule of *M* such that $N \subseteq N_1 \cup ... \cup N_n$. Assume that at most two of the N_k's are not graded primary, and that $Gr(N_j : M) \not\subset Gr(N_k : M)$ whenever $j \neq k$. Then $N \subseteq N_k$ for some *k*.

Proof . We may assume that the covering is efficient since the hypothesis remains valid after reduction to an efficient covering. Then $n \neq 2$. Also $n \leq 2$ by Proposition 2.6. Hence $n = 1$.

Theorem 2.7 [The Graded Prime Avoidance Theorem] Let *M* be a graded moddule over a *G*-graded ring *R*, $N_1,..., N_n$ a finite number of graded submodules of *M* and *N* a graded submodule of *M* such that $N \subseteq N_1 \cup ... \cup N_n$. Assume that at most two of the N_k's are not graded prime, and that $(N_j : M) \not\subset (N_k : M)$ whenever $j \neq k$. Then $N \subseteq N_k$ for some *k*.

Proof. By [1, Proposition 2.3], a graded submodule of a graded module over a *G*-graded ring *R* is prime if and only if it is prime in the graded sense. Now the assertion follows from [2, Theorem 2.3].

Theorem 2.8 [The Graded Primary Avoidance Theorem] Let *R* a *G*-graded ring *R*, $I_1,..., I_n$ a finite number of graded ideals of *R* and *I* a graded ideal of *R* such that $I \subseteq I_1 \cup ... \cup I_n$. Assume that at most two of the I_k's are not graded primary, and that $Gr(I_j) \not\subset Gr(I_k)$ whenever $j \neq k$. Then $I \subseteq I_k$ for some *k*.

Proof .

This follows from Theorem 2.6.

Lemma 2.9

Assume that *R* is a *G*-graded ring and let *N,K* be graded submodules of a graded *R*-module *M* with $K \subseteq N$. Then *N* is a graded primary submodule of *M* if and only if is a graded primary submodule of *M/K*.

Proof. Let *N* be a graded primary submodule of *M* and $r_g(m_h + K) \in N/K$ with $m_h + K \notin N/K$ where $r_g \in h(R)$ and \$ $m_h \in h(M)$. It follows that $r_g m_h \in N$ with $m_h \notin N$, so hence *N* graded primary gives $r_g^s M \subseteq N$ for some *s*; hence $r_g^s(M/K) \subseteq N/K$. Thus *M/K* is graded primary. The remaining implication is similar.

Theorem 2.10 Let *N* be a graded submodule of a graded module *M* over a *G*-graded ring *R*. If the graded primary avoidance theorem hold for *M*, then the graded primary avoidance theorem holds for *M / N*.

Proof. This follows from Lemma 2.9.

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