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Contributed Paper

Infinitesimal Automorphisms in the Tangent Bundle of a Riemannian Manifold with Horizontal Lift of Affine Connection

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ABSTRACT

The main purpose of the present paper is to study conditions for a vertical infinitesimal affine transformation in the tangent bundle of a Riemannian manifold with respect to the horizontal lift of affine connection and then to apply the results obtained to the study of fibre-preserving infinitesimal affine transformation and also to investigate infinitesimal isometry in this setting.

Keywords: lift, tangent bundle, infinitesimal affine transformation, fibre-preserving transformation, infinitesimal isometry.

1. INTRODUCTION

Let M_n be a Riemannian manifold with metric g whose components in a coordinate neighborhood U are g_{ji} and denote by Γ^h_{ji} the Christoffel symbols formed with g_{ji}. If, in the neighborhood $\pi^{-1}(U)$ of the tangent bundle $T(M_n)$ over M_n, U being a neighborhood of M_n, then ${}^H g$ has components given by

$$^H g = \begin{pmatrix} \Gamma^t_j g_{ti} + \Gamma^t_i g_{jt} & g_{ji} \\ g_{ji} & 0 \end{pmatrix}$$

with respect to (x^i, y^i) induced coordinates in $T(M_n)$ and $\Gamma^h_i = y^j \Gamma^h_{ji}$, Γ^h_{ji} being components of the affine connection in M_n.

Let be a pseudo-Riemannian metric, then the horizontal lift ${}^H g$ of g with respect to ∇ is a pseudo-Riemannian metric in $T(M_n)$. Since ${}^H g$ is defined by ${}^H g = {}^C g - \gamma(\nabla g)$, where $\gamma(\nabla g)$ is a tensor field of type , which has the components of the form $\gamma(\nabla g) = \begin{pmatrix} y^s \nabla_s g & 0 \\ 0 & 0 \end{pmatrix}$, we have ${}^H g$ and ${}^C g$ coincide if and only if $\nabla g = 0$ [1, p.105].

If we write $ds^2 = g_{ji} dx^j dx^i$ the pseudo-Riemannian metric in M_n given by g, then the pseudo-Riemannian metric in $T(M_n)$ given by the ${}^H g$ of g to $T(M_n)$ with respect to an affine connection ∇ in M_n is

$$ds^2 = 2 g_{ji} \tilde{\delta} y^j dx^i, \tag{1}$$

where $\tilde{\delta} y^j = dy^j + \tilde{\Gamma}^j_{lk} y^l dx^k$ and $\tilde{\Gamma}^h_{ji} = \Gamma^h_{ij}$ are components of the connection $\tilde{\nabla}$ defined by

$\tilde{\nabla}_X Y = \nabla_Y X + [X,Y], \forall X,Y \in T_0^1(M_n)$, [1, p.67].

We shall now define the horizontal lift ${}^H\nabla$ of affine connection ∇ in M_n to $T(M_n)$ by the conditions

$$
\begin{aligned}
&{}^H\nabla_{{}^VX}{}^VY = 0, && {}^H\nabla_{{}^VX}{}^HY = 0,\\
&{}^H\nabla_{{}^HX}{}^VY = (\nabla_X Y)^V, && {}^H\nabla_{{}^HX}{}^HY = (\nabla_X Y)^H
\end{aligned}
\qquad (2)
$$

for $X,Y \in \Im_0^1(M_n)$. From (2), the horizontal lift ${}^H\nabla$ of ∇ has components ${}^H\Gamma_{JI}{}^K$ such that

$$
{}^H\Gamma_{ij}^k = \Gamma_{ij}^k,\ {}^H\Gamma_{i\bar{j}}^k = {}^H\Gamma_{\bar{i}j}^k = {}^H\Gamma_{\bar{i}\bar{j}}^k = {}^H\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = 0,\ {}^H\Gamma_{ij}^{\bar{k}} = y^s\partial_s\Gamma_{ij}^k - y^s R_{sij}{}^k,\ {}^H\Gamma_{\bar{i}j}^{\bar{k}} = {}^H\Gamma_{i\bar{j}}^{\bar{k}} = \Gamma_{ij}^k, \qquad (3)
$$

with respect to the induced coordinates in $T(M_n)$, where $\Gamma_{ij}{}^k$ are components of ∇ in M_n.

Let g and ∇ be, respectively, a pseudo-Riemannian metric and an affine connection such that $\nabla g = 0$. Then ${}^H\nabla {}^H g = 0$, where ${}^H g$ is a pseudo-Riemannian metric. The connection ${}^H\nabla$ has nontrivial torsion even for the Riemannian connection ∇ determined by g, unless g is locally flat [1, p.111].

Let there be given an affine connection ∇ and a vector field $X \in \Im_0^1(M_n)$. Then the Lie derivative $L_X\nabla$ with respect to X is, by definition, an element of $\Im_2^1(M_n)$ such that

$$
(L_X\nabla)(Y,Z) = L_X(\nabla_Y Z) - \nabla_Y(L_X Z) - \nabla_{[X,Y]}Z = [L_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \qquad (4)
$$

for any $Y,Z \in \Im_0^1(M_n)$.

In a manifold M_n with affine connection ∇, an infinitesimal affine transformation $x^{h'} = x^h + X^h(x^1,...,x^n)\Delta t$ defined by a vector field $X \in \Im_0^1(M_n)$ is called an infinitesimal affine transformation if $L_X\nabla = 0$, [1, p.67].

The main purpose of the present paper is to study the infinitesimal affine transformation and infinitesimal isometry in $T(M_n)$ with affine connection ${}^H\nabla$.

2. VERTICAL INFINITESIMAL AFFINE TRANSFORMATIONS IN A TANGENT BUNDLE WITH ${}^H\nabla$

From (4) we see that, in terms of components $\Gamma_{\gamma\beta}^\alpha$ of ∇, X is an infinitesimal affine transformation in n- dimensional manifold M_n if and only if,

$$
\partial_\gamma\partial_\beta X^\alpha + X^\lambda\partial_\lambda\Gamma_{\gamma\beta}^\alpha - \Gamma_{\gamma\beta}^\lambda\partial_\lambda X^\alpha + \Gamma_{\lambda\beta}^\alpha\partial_\gamma X^\lambda + \Gamma_{\gamma\lambda}^\alpha\partial_\beta X^\lambda = 0,\ \alpha,\beta,... = 1,...,n. \qquad (5)
$$

Let there be given in M_n with a affine connection ∇ with Christoffel symbols Γ_{ij}^k.

Let $\tilde{X} = \tilde{X}^i \partial_i + \tilde{X}^{\bar{i}} \partial_{\bar{i}}$, where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i} = \frac{\partial}{\partial x^{\bar{i}}}$, $\bar{i} = n+1,...,2n$ be a vector field in $T(M_n)$. Then, taking account of (3), we can easily see from (5) that $\tilde{X}$ is an infinitesimal affine transformations in $T(M_n)$ with ${}^H\nabla$ if and only if the following conditions (6)-(13) hold:

$$\partial_j \partial_i \tilde{X}^h + \tilde{X}^k \partial_k \Gamma^h_{ji} - (\Gamma^k_{ji} \partial_k \tilde{X}^h + \partial \Gamma^k_{ji} \partial_{\bar{k}} \tilde{X}^h) + \Gamma^h_{ki} \partial_j \tilde{X}^k + \Gamma^h_{jk} \partial_i \tilde{X}^k + y^s R^k_{sji} \partial_{\bar{k}} \tilde{X}^h = 0, \quad (6)$$

$$\partial_j \partial_{\bar{i}} \tilde{X}^h - \Gamma^k_{ji} \partial_{\bar{k}} \tilde{X}^h + \Gamma^h_{jk} \partial_{\bar{i}} \tilde{X}^k = 0, \quad (7)$$

$$\partial_{\bar{j}} \partial_i \tilde{X}^h - \Gamma^k_{ji} \partial_{\bar{k}} \tilde{X}^h + \Gamma^h_{ki} \partial_{\bar{j}} \tilde{X}^k = 0, \quad (8)$$

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{X}^h = 0, \quad (9)$$

$$\partial_j \partial_i \tilde{X}^{\bar{h}} + (\tilde{X}^k \partial_k \partial \Gamma^h_{ji} + \tilde{X}^{\bar{k}} \partial_k \Gamma^h_{ji}) - (\Gamma^k_{ji} \partial_k \tilde{X}^{\bar{h}} + \partial \Gamma^k_{ji} \partial_{\bar{k}} \tilde{X}^{\bar{h}}) + (\partial \Gamma^h_{ki} \partial_j \tilde{X}^k + \Gamma^h_{ki} \partial_j \tilde{X}^{\bar{k}}) +$$
$$+(\partial \Gamma^h_{jk} \partial_i \tilde{X}^k + \Gamma^h_{jk} \partial_i \tilde{X}^{\bar{k}}) - \tilde{X}^{\bar{k}} R^h_{kji} - y^s \tilde{X}^k \partial_k R^h_{sji} + y^s R^k_{sji} \partial_{\bar{k}} \tilde{X}^{\bar{h}} - y^s R^h_{ski} \partial_j \tilde{X}^k - y^s R^h_{sjk} \partial_i \tilde{X}^k = 0 \quad (10)$$

$$\partial_j \partial_{\bar{i}} \tilde{X}^{\bar{h}} + \tilde{X}^k \partial_k \Gamma^h_{ji} - \Gamma^k_{ji} \partial_{\bar{k}} \tilde{X}^{\bar{h}} + \Gamma^h_{ki} \partial_j \tilde{X}^k + (\partial \Gamma^h_{jk} \partial_{\bar{i}} \tilde{X}^k + \Gamma^h_{jk} \partial_{\bar{i}} \tilde{X}^{\bar{k}}) - y^s R^h_{sjk} \partial_{\bar{i}} \tilde{X}^k = 0, \quad (11)$$

$$\partial_{\bar{j}} \partial_i \tilde{X}^{\bar{h}} + \tilde{X}^k \partial_k \Gamma^h_{ji} - \Gamma^k_{ji} \partial_{\bar{k}} \tilde{X}^{\bar{h}} + (\partial \Gamma^h_{ki} \partial_{\bar{j}} \tilde{X}^k + \Gamma^h_{ki} \partial_{\bar{j}} \tilde{X}^{\bar{k}}) + \Gamma^h_{jk} \partial_i \tilde{X}^k - y^s R^h_{ski} \partial_{\bar{j}} \tilde{X}^k = 0, \quad (12)$$

$$\partial_{\bar{j}} \partial_{\bar{i}} \tilde{X}^{\bar{h}} - \Gamma^h_{ki} \partial_{\bar{j}} \tilde{X}^k + \Gamma^h_{jk} \partial_{\bar{i}} \tilde{X}^k = 0, \quad (13)$$

Let $\tilde{X}$ be a vertical infinitesimal affine transformation in $T(M_n)$. Then $\tilde{X}$ has components $\begin{pmatrix} 0 \\ \tilde{X}^{\bar{h}} \end{pmatrix}$ with respect to the induced coordinates. Thus, from (13), we have $\partial_{\bar{j}} \partial_{\bar{i}} \tilde{X}^{\bar{h}} = 0$, i.e.,

$$\tilde{X}^{\bar{h}} = C^h_i y^i + D^h, \quad (14)$$

where C^h_i and D^h depend only on variables $\tilde{X}^h$. Since $\tilde{X}$ is a vector field in $T(M_n)$, $C = C^h_i \partial_h \otimes dx^i$ and $D = D^h \partial_h$ are defined elements of $\Im^1_1(M_n)$ and $\Im^1_0(M_n)$, respectively.

Theorem 1. If $\tilde{X}$ is a vertical infinitesimal affine transformation of $T(M_n)$ with ${}^H\nabla$, then

(a) $L_D \nabla + \mathrm{C}(D \otimes R) = 0$, $D = \partial^h \frac{\partial}{\partial x^h}$, $D \in \Im^1_0(M_n)$ and $\mathrm{C}(D \otimes R) = D^k R_{kji}{}^h$.

(b) C is parallel with respect to ∇, i.e., $\nabla C = 0$

(c) $C(T(Y,Z)) = T(CY,Z) = T(Y,CZ)$, for any $Y,Z \in \Im_0^1(M_n)$, where T denotes the torsion tensor of ∇, i.e. T is pure tensor with respect to C.

(d) $C(\nabla_Z T)(Y,W) = (\nabla_{CZ} T)(Y,W)$, for any $Y,Z,W \in \Im_0^1(M_n)$.

(e) Conversely, if C and D satisfy the conditions (a), (b), (c) and (d), the vector field

$$\tilde{X} = (C_i^k y^i + D^k)\frac{\partial}{\partial y^k} = \gamma C + {}^v D$$

is an infinitesimal affine transformation of $T(M_n)$ with connection ${}^H\nabla$, where γC is a vertical vector field, which has components of the form $\gamma C = \begin{pmatrix} 0 \\ y^i C_i^h \end{pmatrix}$.

Proof: (a) Substituting (14) and $\tilde{X}^h = 0$ in (10), we have

$$\partial_j \partial_i C_s^h + C_s^k \partial_k \Gamma_{ji}^h - \Gamma_{ji}^k \partial_k C_s^h - \partial_s \Gamma_{ji}^k C_k^h + \Gamma_{ki}^h \partial_j C_s^k + \Gamma_{jk}^h \partial_i C_s^k - C_s^k R_{kji}^h + R_{sji}^k C_k^h = 0, \quad (15)$$

and

$$\partial_j \partial_i D^h + D^k \partial_k \Gamma_{ji}^h - \Gamma_{ji}^k \partial_k D^h + \Gamma_{ki}^h \partial_j D^k + \Gamma_{jk}^h \partial_i D^k - D^k R_{kji}^h = 0, \quad (16)$$

which means that $L_D \nabla + \mathrm{C}(D \otimes R) = 0$.

(b) Substituting (14) $\tilde{X}^h = 0$ and in (12), we obtain,

$$\partial_i C_j^h - \Gamma_{ji}^k C_k^h + \Gamma_{ki}^h C_j^k = 0, \quad (17)$$

Substituting (14) and $\tilde{X}^h = 0$ in (11), we obtain,

$$\partial_j C_i^h - \Gamma_{ji}^k C_k^h + \Gamma_{jk}^h C_i^k = 0, \quad (18)$$

which means C is parallel in M_n.

(c) Interchanging i and j in (18), we have,

$$\partial_i C_j^h - \Gamma_{ij}^k C_k^h + \Gamma_{ik}^h C_j^k = 0,$$

and subtracting the resulting equation from (17), we have,

$$T_{ji}^k C_k^h = T_{ki}^h C_j^k, \quad (19)$$

that is,

$$C(T(Y,Z)) = T(CY,Z) \quad (20)$$

for any $Y,Z \in \Im_0^1(M_n)$. From (19), we obtain $T(Y,CZ)) = -T(CZ,X) = C(T(Z,Y)) = C(T(Y,Z))$ and hence

$$C(T(Y,Z)) = T(CY,Z) = T(Y,CZ)$$

which is the formula (c).

(d) Using (17) and (18), we eliminate all partial derivatives of C_j^h from (15). Then we obtain,

$C_k^h \nabla_j T_{li}^k = \nabla_k T_{li}^h C_j^k$, i.e. T is ϕ - tensor with respect to C [3]. (21)

(e) If we assume that the conditions (a), (b), (c) and (d) are established, then we see that $\tilde{X}$, given in (e), is an infinitesimal affine transformation. Consequently, Theorem 1 is completely proved.

Theorem 2. Let C be as in Theorem 1. If X is an infinitesimal affine transformation of M_n with affine connections ∇ and $R(X,Y,Z;\xi)$ is pure with respect to X and ξ, so is CX.

3. FIBRE-PRESERVING INFINITESIMAL AFFINE TRANSFORMATION WITH $^H\nabla$

A transformation of $T(M_n)$ is said to be fibre-preserving if it sends each fibre of $T(M_n)$ into a fibre. An infinitesimal transformation of $T(M_n)$ is said to be fibre-preserving if it generates a local 1-parameter group of fibre-preserving transformations. An infinitesimal transformation $\tilde{X}$ with components $\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{pmatrix}$ is fibre-preserving if and only if $\tilde{X}^h (h = 1, 2, \ldots, n)$ depend only on the variables $x^1, \ldots, x^n$ with respect to the induced coordinates (x^h, y^h) in $T(M_n)$. From

$$\begin{cases} x^{h'} = x^h + \tilde{X}^h(x^1, \ldots, x^n)\Delta t \\ x^{\bar{h}'} = x^{\bar{h}} + \tilde{X}^{\bar{h}}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n})\Delta t \end{cases}$$

we see that a fibre-preserving infinitesimal transformation $\tilde{X}$ with components $\begin{pmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{pmatrix}$ induces an infinitesimal transformation X with components $\tilde{X}^h$ in the base space M_n. Since $\partial\Gamma_{ji}^k \partial_{\bar{k}} \tilde{X}^h = 0$ and $y^s R_{sji}^k \partial_{\bar{k}} \tilde{X}^h = 0$, then from (6) we have:

Theorem 3. If $\tilde{X}$ a fibre-preserving infinitesimal transformation of $T(M_n)$ with horizontal lift $^H\nabla$ of a affine connection ∇ in M_n to $T(M_n)$, then the infinitesimal transformation X induced on M_n from $\tilde{X}$ is also affine with respect to ∇.

Theorem 4. Let ∇ be a affine connection in M_n. Then,

$$(L_{^C X} {}^H\nabla)(^C Y, ^C Z) = ^C (L_X \nabla)(^C Y, ^C Z) + \gamma(L_X R)(\ , Y, Z), \text{ for any } X \in \Im_0^1(M_n).$$

Proof. Our proposition follows from the following computations:

$$(L_{{}^C X}{}^H\nabla)({}^C Y, {}^C Z) = L_{{}^C X}({}^H\nabla_{{}^C Y}{}^C Z) - {}^H\nabla_{{}^C Y}(L_{{}^C X}{}^C Z) - {}^H\nabla_{[{}^C X, {}^C Y]}{}^C Z$$

$$= L_{{}^C X}[{}^C(\nabla_Y Z) - \gamma(R(\ ,Y)Z)] - {}^H\nabla_{{}^C Y}{}^C(L_X Z) - {}^H\nabla_{{}^C[X,Y]}{}^C Z$$

$$= [{}^C X, {}^C\nabla_X Y] - [{}^C X, \gamma(R(\ ,Y)Z)] - {}^C(\nabla_Y(L_X Z)) + \gamma(R(\ ,Y)L_X Z)$$

$$- {}^C(\nabla_{[X,Y]}Z) + \gamma R([X,Y]Z)]$$

$$= {}^C(L_X\nabla_X Y) - {}^C(\nabla_Y(L_X Z)) - {}^C(\nabla_{[X,Y]}Z) - \gamma(L_X R(\ ,Y)Z)$$

$$+ \gamma(R(\ ,Y)L_X Z) + \gamma(R(\ ,L_X Y)Z)$$

$$= {}^C(L_X\nabla)({}^C Y, {}^C Z) + \gamma(-L_X R(\ ,Y)Z + R(\ ,Y)L_X Z + R(\ ,L_X Y)Z)$$

$$= {}^C(L_X\nabla)({}^C Y, {}^C Z) - \gamma(L_X R)(\ ,Y,Z)$$

where $R(\ ,X)Y$ denotes a tensor field W of type (1,1) in M_n such that $W(Z) = R(Z,X)Y$ for any $Z \in \Im_0^1(M_n)$. Let $\tilde{X}$ and X be as in Theorem 3. From Theorem 4 we see that, it is also known that if X is infinitesimal affine transformation, $L_X R = 0$ [5], then ${}^c X$ is an infinitesimal affine transformation of $T(M_n)$ with ${}^H\nabla$. Since ${}^c X$ has the components $\begin{pmatrix} X^h \\ \partial X^h \end{pmatrix}$, it follows that $\tilde{X}$ - ${}^c X$ is a vertical infinitesimal affine transformation in $T(M_n)$ with ${}^H\nabla$. Thus we have,

Theorem 5. If $\tilde{X}$ is a fibre-preserving infinitesimal affine transformation of $T(M_n)$ with the lift ${}^H\nabla$, then $\tilde{X} = {}^c X + {}^v D + \gamma C$, where D and C are tensor fields of type $(1,0)$ and $(1,1)$, respectively, satisfying conditions (a), (b) and (c) of Theorem 1.

4. INFINITESIMAL ISOMETRY WITH ${}^H g$

A vector field $X \in \Im_0^1(M_n)$ is said to be an infinitesimal isometry or a Killing vector field of a Riemannian manifold with metric g, if $L_X g = 0$ [4]. In terms of components g_{ij} of g, X is infinitesimal isometry if and only if

$$L_X g_{ij} = X^\alpha \nabla_\alpha g_{ij} + g_{\alpha j}\nabla_i X^\alpha + g_{i\alpha}\nabla_j X^\alpha = \nabla_j X_i + \nabla_i X_j$$

X^α being components of X, where ∇ is the Riemannian connection of the metric g .

Let $\tilde{X}$ be vector field in $T(M_n)$ and $(\tilde{X}^A) = \begin{pmatrix} \tilde{X}^h \\ \tilde{X}^{\bar{h}} \end{pmatrix}$ its components with respect to induced coordinates. Then the covariant derivative ${}^H\nabla\tilde{X}$ has components

$$ {}^{H}\nabla_I \tilde{X}^J = \partial_I X^J + {}^{H}\Gamma^J_{IM} \tilde{X}^M \tag{22} $$

${}^{s}\Gamma^J_{IM}$ being given by (3), with respect to induced coordinates.

We now consider a vector field $X \in \Im^1_0(M_n)$, then its vertical lift ${}^{V}X \in \Im^1_0(T(M_n))$, complete lift ${}^{c}X \in \Im^1_0(T(M_n))$ and horizontal lift ${}^{H}X \in \Im^1_0(T(M_n))$ have respectively components of the form

$$ {}^{V}X = \begin{pmatrix} 0 \\ x^h \end{pmatrix},\ {}^{c}X = \begin{pmatrix} x^h \\ \partial x^h \end{pmatrix},\ {}^{H}X = \begin{pmatrix} x^h \\ -\Gamma^h_i x^i \end{pmatrix} \tag{23} $$

with respect to the induced coordinates in $T(M_n)$, where $\Gamma^h_i x^i = y^s \Gamma^h_{si} x^i$.

We now compute the Lie derivatives of the metric ${}^{H}g$ with respect to ${}^{V}X$ ${}^{c}X$, and ${}^{H}X$, by means of (3) and (23). The Lie derivatives of ${}^{H}g$ with respect to ${}^{V}X$, ${}^{c}X$ and ${}^{H}X$ have respectively components

$$ \begin{cases} L_{{}^{V}X}\, {}^{H}g = ({}^{H}\nabla_I\, {}^{V}X^J + {}^{H}\nabla_J\, {}^{V}X^I) = \begin{pmatrix} 0 & 0 \\ \nabla_j X^i + \nabla_i X^j & 0 \end{pmatrix} \\ L_{{}^{c}X}\, {}^{H}g = ({}^{H}\nabla_I\, {}^{c}X^J + {}^{H}\nabla_J\, {}^{c}X^I) = \begin{pmatrix} \nabla_i X^j + \nabla_j X^i & 0 \\ y^s \partial_s (\nabla_j X^i + \nabla_i X^j) - y^s (R_{sik}{}^{j} + R_{sjk}{}^{i}) X^k & \nabla_i X^j + \nabla_j X^i \end{pmatrix} \\ L_{{}^{H}X}\, {}^{H}g = ({}^{H}\nabla_I\, {}^{H}X^J + {}^{H}\nabla_J\, {}^{H}X^I) = \begin{pmatrix} \nabla_i X^j + \nabla_j X^i & 0 \\ -\Gamma^j_h \nabla_i X^h - \Gamma^i_h \nabla_j X^h & \nabla_i X^j + \nabla_j X^i \end{pmatrix} \end{cases} \tag{24} $$

Taking account of the fact that $\nabla_i X^k = 0$ implies $R_{sik}{}^{j} X^k = 0$ and $R_{sjk}{}^{i} X^k = 0$
We have,

Theorem 6. Necessary and sufficient conditions in order that

a) Vertical ${}^{V}X \in \Im^1_0(T(M_n))$

b) Complete ${}^{c}X \in \Im^1_0(T(M_n))$

c) Horizontal ${}^{H}X \in \Im^1_0(T(M_n))$

lifts to $T(M_n)$ with the metric ${}^{H}g$, of a vector field X in M_n be a Killing vector field in $T(M_n)$ are that

a) X is infinitesimal isometry in M_n

b) X is infinitesimal isometry in M_n with vanishing covariant derivation in M_n.

c) X is infinitesimal isometry in M_n with vanishing covariant derivation in M_n.

Let X and Y be vector fields in M_n. If X and Y are Killing vector fields in M_n, from the definition of Killing vector field, then we have,

$$L_{[x,y]}g = L_X(L_Y g) - L_Y(L_X g) = 0 \tag{25}$$

i.e. $[X,Y]$ is infinitesimal isometry in M_n [3].

We now denote by $A_X Y$ the tensor field of type (1,1) , X and Y being two given elements of $\Im_0^1(M_n)$, defined by

$$(A_X Y)Z = (L_X \nabla)(Y,Z) = [L_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \tag{26}$$

for any $Z \in \Im_0^1(M_n)$. Then we have,

$$[{}^V X, {}^c Y] = {}^V[X,Y], [{}^c X, {}^c Y] = {}^c[X,Y], [{}^c X, {}^H Y] = {}^H[X,Y] - \gamma(A_X Y). \tag{27}$$

where $\gamma(A_X Y) \in \Im_0^1(M_n)$, which has the components of the form $\begin{pmatrix} 0 \\ y^s (A_X Y)_s^h \end{pmatrix}$. An infinitesimal transformation defined by vector field $X \in \Im_0^1(M_n)$ is said to be infinitesimal transformation with affine connection ∇, if $L_X \nabla = 0$. Then, from (26) and (27)

$$[{}^c X, {}^H Y] = {}^H[X,Y] \tag{28}$$

We compute the Lie derivatives of the metric ${}^H g$ with respect to ${}^V[X,Y]$ and ${}^c[X,Y]$

$$L_{{}^V[X,Y]}{}^H g = L_{[{}^V X, {}^c Y]}{}^H g = L_{{}^V X}(L_{{}^c Y}{}^H g) - L_{{}^c Y}(L_{{}^V X}{}^H g)$$

$$L_{{}^c[X,Y]}{}^H g = L_{[{}^c X, {}^c Y]}{}^H g = L_{{}^c X}(L_{{}^c Y}{}^H g) - L_{{}^c Y}(L_{{}^c X}{}^H g) \tag{29}$$

from (24) and (29), we get,

Theorem 7. Sufficient conditions in order that the vertical, complete lifts of a vector field $[X,Y]$ in M_n to $T(M_n)$ be a infinitesimal isometry with metric ${}^H g$ are that X and Y is a infinitesimal isometry in M_n with vanishing their covariant derivations in M_n.

Let X be infinitesimal affine transformation in M_n. From (25) and (29), we have

$$L_{{}^H[X,Y]}{}^H g = L_{[{}^c X, {}^H Y]}{}^H g = L_{{}^c X}(L_{{}^H Y}{}^H g) - L_{{}^H Y}(L_{{}^c X}{}^H g) \tag{30}$$

from (24) and (30), we get,

Theorem 8. Sufficient conditions in order that the horizontal lift of a vector field $[X,Y]$ in M_n to $T(M_n)$ be a infinitesimal isometry with metric ${}^H g$ are that X and Y are infinitesimal isometry with vanishing their covariant derivations in M_n.

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