



Nonlinear Model-Based Control for Parabolic Partial Differential Equations Systems

Atthasit Tawai, Chanin Panjapornpon*

Department of Chemical Engineering, Faculty of Engineering, Kasetsart University, Bangkok 10900, Thailand.

*Author for correspondence; e-mail: fengcnp@ku.ac.th

Received : 17 October 2007

Accepted : 30 October 2007

ABSTRACT

This work proposed the extended technique of nonlinear feedback controller in [2] for nonlinear distributed parameter system governed by parabolic partial differential equations (PDEs). The concepts of modal decomposition and Galerkin projection under the assumption of discrete eigenspectrum in [7] are used to approximate the model in the form of ordinary differential equations (ODEs). The combination of modal decomposition analysis with the control method in [2] is applied to a typical transport-reaction process with multiple steady states. The simulation results showed that the controller is successfully employed to control the temperature profiles of the process at desired output.

Keywords: Nonlinear model-based control, parabolic PDEs, modal analysis, transport-reaction processes, approximate input-state linearization.

1. INTRODUCTION

Over a decade, key technological needs in bloom areas such as nanotechnology and biotechnology have motivated many researches to work on analysis and control of complex systems across all engineering disciplines. Many chemical and biochemical engineering processes for example tubular reactors and crystal growth are naturally complex distributed processes in which the state space of the processes is infinite-dimensional with high nonlinearity systems. The regulation of spatial profiles in processes coupled with transport and reaction phenomena is difficult due to the high nonlinearities which can be modeled by partial differential equations (PDEs).

To solve these spatial distributions, simplifications of these transport and kinetic

properties in experiments with the assumption of the lumped parameter model were proposed in many works. In conventional approaches, the mathematical model of distributed parameter systems (DPS) was simplified by discretization technique involving the use of finite difference/finite element techniques. Modal analysis is an attractive method of treating PDEs. It is the natural reduction technique and works well with only a few modes if the eigenvalues are not bunched together. However, this approach leads to approximated systems with too many ordinary differential equations which are not suitable for dynamical analysis, controller synthesis or real-time controller implementation.

The extended technique of nonlinear

feedback controllers in [2] for distributed system described by parabolic PDEs that can be applied to both stable and unstable process is proposed in this work. The combination of modal decomposition analysis to approximate the PDEs model with the consideration through Galerkin projection under the assumption of discrete eigen spectrum in [7] is proposed to reduce the number of the ODE equations which is used to describe the process dynamics. An application of proposed method is illustrated in a transport-reaction process which is described by unstable parabolic PDE system.

This paper is organized as follows. The mathematical models, control method and an application will be proposed briefly in section 2. Section 3 shows and discusses the results from the simulation of closed-loop response. Finally, the conclusions are given in section 4.

2. MATERIALS AND METHODS

2.1 Description of parabolic system

In this work, we simply consider parabolic PDEs systems in one spatial dimension with a state-space description of the form:

$$\begin{aligned} \frac{\partial x}{\partial t} &= A \frac{\partial x}{\partial z} + B \frac{\partial^2 x}{\partial z^2} + wb(z)u + f(x) \\ y^i &= \int_{\alpha}^{\beta} c^i(z) x dz \quad i=1, \dots, l \end{aligned} \tag{1}$$

subject to the boundary and initial conditions

$$\begin{aligned} C_1 x(\alpha, t) + D_1 \frac{dx}{dz}(\alpha, t) &= R_1, \quad C_2 x(\beta, t) + D_2 \frac{dx}{dz}(\beta, t) = R_2 \\ x(z, 0) &= x_0(z), \quad u_i \leq u \leq u_{hi}, \quad i=1, \dots, m \end{aligned} \tag{2}$$

where $x(z, t) = [x_1(z, t) \dots x_n(z, t)]^T$ denotes the vector of state variables; $z \in [\alpha, \beta]$ is the spatial coordinate; $u = [u^1 \ u^2 \ \dots \ u^l]^T \in R^l$ denotes the vector of manipulated inputs; $y = [y_1 \ \dots \ y_m]^T \in R^m$ denotes the vector of controlled outputs; $w, k, A, B, C_p, C_2, D_p, D_2, R_l$ and R_m are constant matrices. The vector $b(z)$ is a known smooth function of $z, b(z) = [b^1(z) \ b^2(z) \ \dots \ b^l(z)]$, where $b^i(z)$ describes how the control actions; $u(t)$ is distributed in the spatial interval $[\alpha, \beta]$ and $c^i(z)$ is known smooth functions.

The system in the form such (1) can be treated by the technique of separation of variables. This equation is amenable to modal decomposition. Thus we assume a solution of the form

$$x(z, t) = \sum_{n=0}^{\infty} a_n(t) \phi_n(z) \tag{3}$$

where $a_n(t)$ denotes the eigencoefficient and $\phi_n(z)$ denotes the corresponding eigenfunction.

We also assume that $u(z, t)$ and nonlinear terms $f(x)$ can be represented in a separable fashion with the same functions $\phi_n(z) : u(z, t) = \sum_{n=0}^{\infty} b_n(t) \phi_n(z)$ and $f(x) = \sum_{n=0}^{\infty} f_n(t) \phi_n(z)$. This will always be possible if $\phi_n(z)$ represent a complete set of basis function. Thus, it is possible to give any distribution of $x(z, t)$ by determination of $a_n(t)$.

Substituting of equation (3), $u(z,t)$ and $f(x)$ in (1) and dividing by $\phi_n(z)a_n(t)$ produces

$$\frac{1}{a_n} \frac{da_n}{dt} = \frac{1}{\phi_n} \frac{d^2\phi_n}{dz^2} + \frac{b_n(t)u(t)}{a_n(t)} + \frac{f_n(t)}{a_n(t)} \quad (4)$$

The equation (4) can be separated into only functions of time and only functions of z as follows

$$\frac{1}{a_n} \frac{da_n}{dt} - \frac{b_n(t)u(t)}{a_n(t)} - \frac{f_n(t)}{a_n(t)} = \lambda_n \quad (5)$$

$$\frac{1}{\phi_n} \frac{d^2\phi_n}{dz^2} = \lambda_n \quad (6)$$

where λ_n is a constant. Let us now rewrite these as:

$$\frac{da_n}{dt} - \lambda_n a_n = b_n(t)u(t) + f_n(t) \quad n = 0, 1, 2, \dots \quad (7)$$

$$\frac{d^2\phi_n}{dz^2} - \lambda_n \phi_n = 0 \quad n = 0, 1, 2, \dots \quad (8)$$

The above equations showed that (7) can be used to determine $a_n(t)$ after the unknown variables, $b_n(t)$, $f_n(t)$ and λ_n , were calculated.

Consider the coefficients $b_n(t)$ and $f_n(t)$ for the series representation of $u(z,t)$ and $f(x)$, they are given by the expression:

$$b_n(t) = \int_{\alpha}^{\beta} \phi_n(z)u(t)dz, \quad f_n(t) = \int_{\alpha}^{\beta} \phi_n(z)f(x)dz \quad (9)$$

The Eq. (8) can be used for calculation of λ_n , together with an application of the boundary conditions as shown in [8]. Consideration through Galerkin projection under the assumption of discrete eigenspectrum in [7], the dimension of the ODE system which equal to the number of slow modes can be obtained. The Eq. (7) is now allows us to calculate the value of $a_n(t)$:

2.2 Control method

The parabolic PDE system is now approximated in the form of (7) which can be recast as:

$$\begin{aligned} \frac{da}{dt} &= f(a,u) & a(0) &= a_0 \\ y &= h(a) \end{aligned} \quad (10)$$

where $a = [a_1 \cdots a_n]^T \in R^n$ is the vector of extended state variables, $u = [u_1 \cdots u_m]^T \in R^m$ is

the vector of manipulated inputs, $y = [y_1 \cdots y_m]^T \in R^m$ is the vector of controlled outputs, $f(a, u) = [f_1(a, u) \cdots f_n(a, u)]^T$ and $h(a) = [h_1(a) \cdots h_m(a)]^T$ are smooth vector functions. The relative order (degree) of a state variable, a_i , is denoted by r_i . The control method in [2] is modified for a simple case with single input single output. For the given output set-point, y_{sp} , the corresponding desired steady-state pair (a_{ss}, u_{ss}) satisfies

$$0 = f(a_{ss}, u_{ss}) \quad \text{and} \quad y_{sp} = h(a_{ss}) \tag{11}$$

These relations are used to describe the dependence of a nominal (desired) steady state, a_{ss} , on the set-point, so $a_{ss} = F(y_{sp})$. The control system in [2] is the form

$$\begin{aligned} \dot{z} &= F_\eta(z + Ly, y, u) - LF_y(z + L, y, u) \\ \hat{a} &= T^{-1}(z + Ly, y) \\ \hat{\eta} &= z + Ly \\ \dot{w} &= f(w, \Psi(w, v)) \\ v &= F(y_{sp} - y + h(w)) \\ u &= \Psi(\hat{a}, v) \end{aligned} \tag{12}$$

where $\Psi(a, v)$ is a nonlinear state feedback described by

$$\Psi(a, v) = \min_u \sum_{i=1}^n \left[\frac{a_i + \sum_{l=1}^{r_i-1} \epsilon_i^l \binom{p_i}{l} H_i^l(a) + \sum_{l=r_i}^{p_i} \epsilon_i^l \binom{p_i}{l} H_i^l(a, u, 0, \dots, 0) - v_i}{\epsilon_i^{p_i}} \right]^2 \tag{13}$$

subject to

$$u_i \leq u \leq u_{p_i}, \quad i = 1, \dots, m$$

where $p_1 \geq r_1, \dots, p_m \geq r_m$, and $\epsilon_1, \dots, \epsilon_n$ are positive constants that set the speed of the state responses in closed-loop. The constant vector L is the observer gain. The nonlinear state feedback is derived by minimizing a function norm of the deviations of the extended state variables from their linear reference trajectories. The speed of the closed-loop extended state responses can be set by adjusting the value of the control system parameter $\epsilon_1^1, \dots, \epsilon_n^1$, the faster of the extended state response can be obtained from smaller value of ϵ_i .

2.3 Application to a transport-reaction process

Consider a long, thin rod in a furnace with the zeroth-order exothermic catalytic reaction of the form $A \rightarrow B$ takes place on the rod as shown in Fig. 1. To control the temperature of the exothermic reactor, cooling water is used by contacting with the rod. The spatiotemporal evolution of the dimensionless rod temperature is described by the following parabolic PDE:

$$\frac{\partial \bar{x}}{\partial t} = B \frac{\partial^2 \bar{x}}{\partial z^2} + \beta_T e^{-\gamma/(1+\bar{x})} + \beta_U (b(z)u(t) - \bar{x}) - \beta_{T,n} e^{-\gamma}$$

$$y(t) = \int_0^\pi \bar{x}(z, t) dz$$
(14)

subject to the Dirichlet boundary conditions and the initial condition:

$$\bar{x}(0, t) = 0, \quad \bar{x}(\pi, t) = 0 \text{ and } \bar{x}(z, 0) = \bar{x}_0(z)$$
(15)

where \bar{x} denotes a dimensionless temperature of the rod; y denotes the controlled output; β_T , $\beta_{T,n}$, γ and β_U denote a dimensionless heat of reaction, a nominal dimensionless heat of reaction, a dimensionless activation energy and a dimensionless heat transfer coefficient, respectively; and u denotes the manipulated input (temperature of the cooling medium). The following typical values were given to the process parameters $\beta_{T,n} = 50$, $\beta_U = 2$, $\gamma = 4$. The system can be decomposed to ODE in form of (7) by chosen as $n=3$. The eigenvalue problem can be solved analytically with the application of the boundary and initial conditions to calculate λ_n and its solution is presented as the form:

$$\lambda_n = -n^2, \quad \phi_n(z) = \sqrt{\frac{2}{\pi}} \sin(n \cdot z)$$
(16)

Although the eigenvalues of the system are all stable, the spatially uniform operating steady state $y(t) = 0$ of the system of (14) is unstable. The controller system for the process will be applied to the process as minimum relative order with $p_1 = 1$, $p_2 = 2$ and $p_3 = 2$, the application takes the form:

$$u = \min_u \left[\left(\frac{-v_1 + \hat{a}_1 + \varepsilon_1 \dot{\hat{a}}_1}{\varepsilon_1^{p_1}} \right)^2 + \left(\frac{-v_2 + \hat{a}_2 + 2\varepsilon_2 \dot{\hat{a}}_2 + \varepsilon_2^2 \ddot{\hat{a}}_2}{\varepsilon_2^{p_2}} \right)^2 + \left(\frac{-v_3 + \hat{a}_3 + 2\varepsilon_3 \dot{\hat{a}}_3 + \varepsilon_3^2 \ddot{\hat{a}}_3}{\varepsilon_3^{p_3}} \right)^2 \right]$$
(17)

The following controller parameter values are used as adjustable parameters: $\varepsilon_1 = 0.001$, $\varepsilon_2 = 0.35$, $\varepsilon_3 = 0.35$, $L_1 = 0.002$, $L_2 = 0.01$ and $L_3 = 0.002$. MATLAB optimization toolbox is used to solve the optimization problem of the controller system.

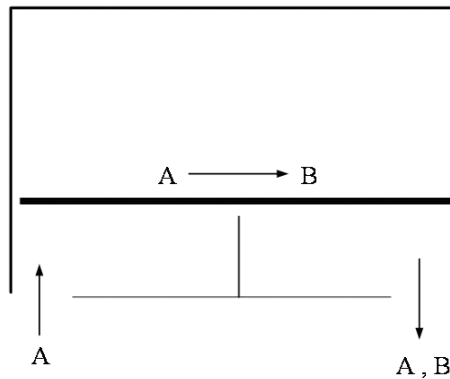


Figure 1 Catalytic rod.

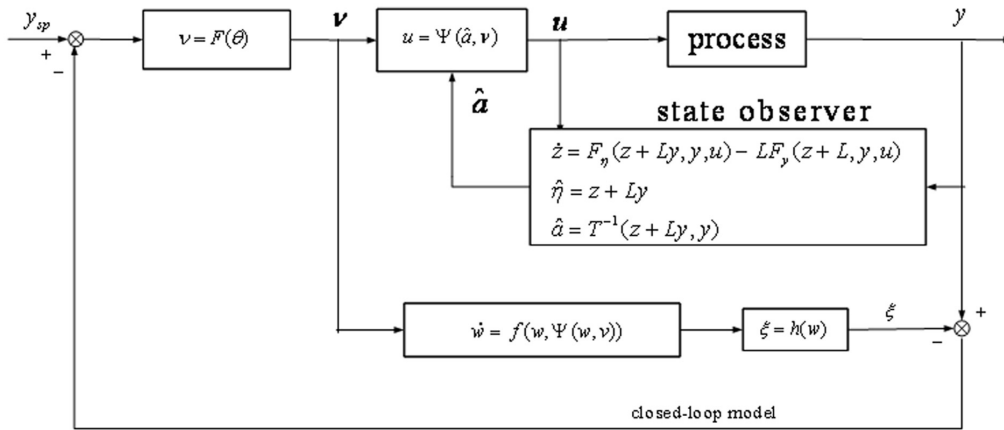


Figure 2 Controller system

3. RESULTS AND DISCUSSION

The open-loop profile of the applied transport-reaction process is shown in Fig. 3; the output moves to another steady-state instead of the desired condition. The constrained optimization problems in (17) are solved numerically to force the states to follow the design trajectories. The closed-loop output response under the controller system is shown as Fig. 4. The profile of manipulated input corresponding to Fig. 4 is shown in Fig. 5. The simulation results show that the controller successfully operates the temperature of the example rod process at desired output, which is open-loop unstable. The controller is

capable of operating the process at the desired steady state, regardless of the initial conditions of the process. In addition, when the process is at the desired set-point, the servo performance of the controller is studied by changing the set point from 0 to 0.1. The output response and manipulated input response of the controller to the servo problem is shown in Fig. 6 and Fig. 7, respectively. It is clear that the controller can force the control variable to move to any condition such the required steady state condition.

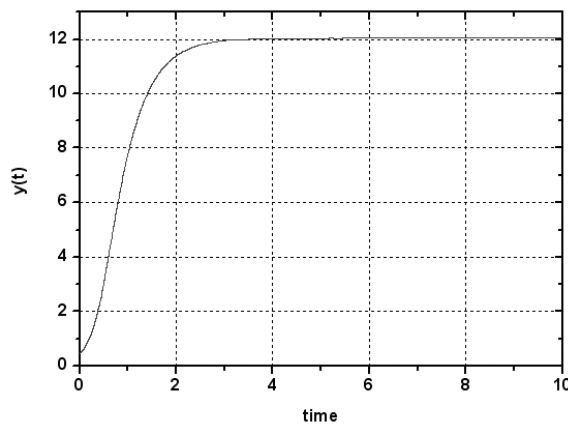


Figure 3 Open-loop profile of $y(t)$ in the catalytic rod.

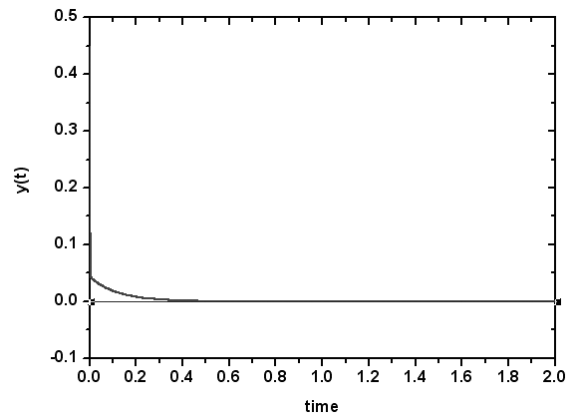


Figure 4 Closed-loop response of $y(t)$.

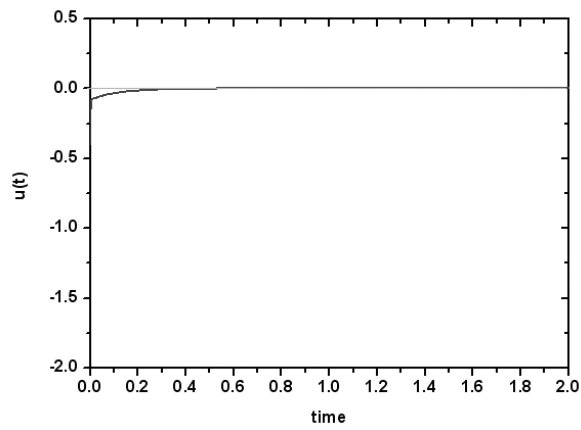


Figure 5 Manipulated input profiles corresponding to Figure 2

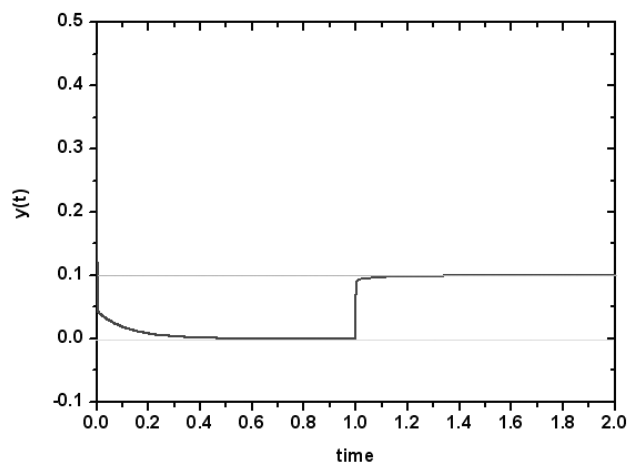


Figure 6 Closed-loop response of $y(t)$ under servo control.

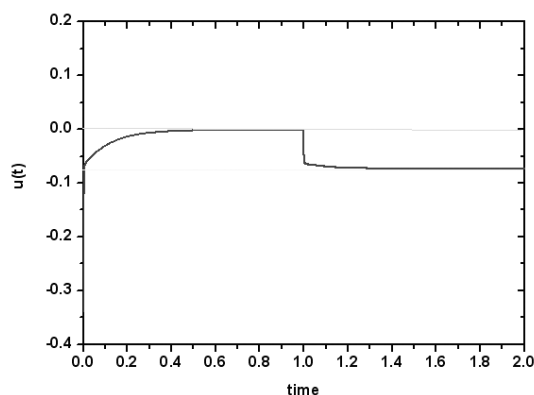


Figure 7 Manipulated input profiles corresponding to Fig. 6 under the servo control.

4. CONCLUSIONS

This work proposed an application of a control methodology to a transport-reaction process described by the parabolic PDE model. The PDE model is extended for approximation of the system to produce an ODE system. The results of solving for steady state condition showed that the example process exhibited multiple steady states, and the output response of open loop system moves to undesired condition. The proposed nonlinear control system based on the approximated input-state linearization along with reduced-order observer successfully forces the output to move through the desired setpoints asymptotically.

ACKNOWLEDGEMENTS

This work was supported by the Faculty of Engineering, Kasetsart University Grants 50/19/CHEM and the National Center of Excellence for Petroleum, Petrochemical and Advanced Materials. These supports are gratefully acknowledged.

REFERENCES

- [1] Christofides, P.D. and Baker, J., Robust output feedback control of quasi-linear parabolic PDE systems. *Systems & Control Letters*, 1999; 36: 307-316.
- [2] Panjapornpon, C. and Soroush, M., Model-Based Controller Design for Unstable, Non-Minimum-Phase, Nonlinear Processes, *Ind. Eng. Chem. Res.*, 2006; 45: 2758-2768.
- [3] Kanter, J.M., Soroush M. Seider W.D., Nonlinear feedback control of multiple non-minimum-phase processes. *J. Proc. Contr.*, 2002; 12: 667-686.
- [4] Niemiec, M. P., Kravaris C., Nonlinear model-state feedback control for non-minimum-phase processes. *Automatica*. 2003; 39(7): 1295-1302.
- [5] Palazoglu, A., Karakas A., Control of nonlinear distributed parameter systems using generalized invariants. *Automatica*, 2000; 36: 697-703.
- [6] Soroush, M., Nonlinear state-observer design with application to reactors. *Chem. Eng. Sci.*, 1997; 52 (3): 387.
- [7] Christofides, P.D., *Nonlinear and Robust Control of PDE systems.*, Berkhauser, Boston, 2001.
- [8] Ray, W. H., *Advanced Process Control*, McGraw-Hill, New York, 1989.