Chiang Mai J. Sci. 2008; 35(2) : 231-240
www.science.cmu.ac.th/journal-science/josci.html
Contributed Paper

Two New Approaches to Peratization Technique

Lalit K. Sharma and Pearson V. Luhanga
Department of Physics, University of Botswana, Private Bag 0022, Gaborone, Botswana.
*Author for correspondence; e-mail: sharmalk@mopipi.ub.bw

Received: 19 October 2007
Accepted: 31 January 2008.

ABSTRACT

The peratization scheme normally used to obtain scattering length $A(\alpha)$ is through the Born series in powers of the coupling constant g. In this paper, starting with the standard equation for the scattering length, two new approaches to peratization technique have been discussed. These approaches have been applied to different potentials successfully.

1. INTRODUCTION

The study of repulsive singular potentials has been the subject of investigation for many years. This study has been applied in a variety of fields such as unrenormalizable field theories [1-4], high energy behaviour of phase shifts [5-8], and low energy behaviour of the phase shifts [9-10] and to some physical applications such as molecular physics [11-12]. Not only that, this problem has relevance in atomic and molecular physics but is also investigated by theoreticians with the hope that its analogy [2, 13-14] with the problem of infinities in both renormalizable and unrenormalizable field theories is actually significant, and also that a better understanding of the potential problem may throw some light on the much more obscure relativistic problem. The similarity between these two problems rests essentially on the fact that both of them yield divergent results, if attacked by the perturbation theory.

For singular potentials it is well- known that the problem is physically reasonable only if the singular core is repulsive and that only in this case the quantities which measure the scattering are defined unambiguously. Thus major work in this field has been primarily devoted to repulsive singular potentials [15-18]. For a singular potential $gV(r)$, the scattering length A considered as a function of coupling constant g, has a singularity at $g = 0$ due to the singular nature of the potential [19]. Hence, a power expansion of A in g is frustrated by infinite integrals. In fact the failure of the perturbation procedure in the case of singular potentials led to the idea of peratization, a technique first introduced by Fienberg and Pais [20] to deal with unrenormalizable field theories. The technique of peratization is designed to give meaning to a series in which each term is a divergent function of a parameter. It has been applied for the calculation of scattering length of various repulsive singular potential [1,15, 21-25].

The success of regularization is presumed to be the precondition for the success of peratization. In this paper, we use the "θ" and

"+" regularizations. The power expansion in g for the scattering length, corresponding to regularized potentialg $V(r, \alpha)$, can be written as

$$A(\alpha) = \sum_{n=1}^{\infty} A_n(\alpha) g^n , \tag{1}$$

where $A_n(\alpha)$ are functions of α which diverge as $\alpha \to 0$. The peratization consists of summing up the series of most singular terms of $A_n(\alpha)$ in each power of g and to see finally whether the sum is finite, when the limit $\alpha \to 0$ is approached. The usual procedure of obtaining the series (1) so far has been through the Born series.

The object of this paper is to formulate two new systematic approaches for obtaining series (1).The formulation of these two approaches has been discussed in sections 2 and 3 respectively.

2. Approach 1

In this new method, we obtain the series (1) by a different approach which is discussed in section 2.1. In sections, 2.2 and 2.3, applications of this new approach to some specific singular potential have been presented.

2.1 Formulation of the Method

Instead of obtaining series (1) by using the Born series, it is assumed that each coefficient $A_n(\alpha)$ in (1) is obtained from the function $A_n(r,\alpha)$ in the limit $r \to \infty$, so that

$$\lim_{r \to \infty} \quad A_n(r,\alpha) = A_n(\alpha) \tag{2}$$

and

$$\lim_{r \to \infty} \qquad A(r,\alpha) = A(\alpha) \tag{3}$$

From (2) and (3), it is found that (1) can be obtained in the limit $r \to \infty$, if one sets

$$A(r,\alpha) = \sum_{n=1}^{\infty} A_n(r,\alpha) g^n , \tag{4}$$

where $A(r,\alpha)$ is the interpolating scattering length corresponding to the regularized potential $gV(r,\alpha)$.

The standard equation for scattering length [7] for the potential $gV(r)$ is given as

$$A'(r) = -gV(r)[r + A(r)]^2 . \tag{5}$$

Substituting (4) in (5) yields

$$\frac{d}{dr}\sum_{n=1}^{\infty} A_n(r,\alpha) g^n = -gV(r,\alpha)\left[r + \sum_{n=1}^{\infty} A_n(r,\alpha) g^n\right]^2 \tag{6}$$

Equating equal powers of g on both sides, one gets the following simple first -order differential equation for $A_n(r,\alpha)$ which can easily be solved, thus

$$A_1'(r,\alpha) = -V(r,\alpha)r^2; \tag{7a}$$

$$A_2'(r,\alpha) = -V(r,\alpha)\{2rA_1(r,\alpha)\}; \tag{7b}$$

$$A_3'(r,\alpha) = -V(r,\alpha)\{2rA_2(r,\alpha) + A_1^2(r,\alpha)\}; \tag{7c}$$

$$A_4'(r,\alpha) = -V(r,\alpha)2\{rA_3(r,\alpha) + A_1(r,\alpha)A_2(r,\alpha)\}, \tag{7d}$$

and so on.

To remove divergence in the results, generally, the following two regularization schemes have been used in the peratization calculations:

(i) "θ" regularization

$$V_\theta(r,\alpha) = V(r)\theta(r-\alpha), \tag{8}$$

where $\theta(x)$ is a step- function, which is unity for the positive values of the argument or zero otherwise. Obviously for "θ" regularisation, the limits of integration will be from $r = \alpha$ to r in (7).

(ii) "$+$"regularization

$$V_+(r,\alpha) = V(r+\alpha), \tag{9}$$

where $V(r)$ is considered in the limits as $\alpha \rightarrow 0^+$ of the sequence. In this case, the limits of integration will be from $r = 0$ to r in (7).

2.2 Application to the Potential $\frac{g}{r^4}$

We now apply above theory to the simplest form of a singular potential i.e.

$$V(r) = \frac{g}{r^4} \tag{10}$$

This potential, attractive or repulsive, has been studied most extensively [26]. Regge behaviour of this potential has also been studied [16-17, 27]. For "θ" regularization of potential (10), equations (7) give,

$$A_1(r,\alpha) = \left\{\frac{1}{r} - \frac{1}{\alpha}\right\}; \tag{11a}$$

$$A_2(r,\alpha) = 2\left\{\frac{1}{3r^3} - \frac{1}{2\alpha r^2} + \frac{1}{6\alpha^3}\right\}; \tag{11b}$$

$$A_3(r,\alpha) = \left\{\frac{7}{15r^5} - \frac{1}{\alpha r^4} + \frac{1}{3\alpha^2 r^3} + \frac{1}{3\alpha^3 r^2} - \frac{2}{15\alpha^5}\right\} \tag{11c}$$

$$A_4(r,\alpha) = 2\begin{Bmatrix} \dfrac{17}{105r^7} - \dfrac{4}{9\alpha r^6} + \dfrac{4}{15\alpha^2 r^5} + \dfrac{1}{6\alpha^3 r^4} \\ -\dfrac{1}{9\alpha^4 r^3} - \dfrac{1}{15\alpha^5 r^2} + \dfrac{17}{630\alpha^7} \end{Bmatrix}, \tag{11d}$$

and so on.

From (2), (3), (4) and (11), we get:

$$A(\alpha) = -\frac{g}{\alpha} + \frac{g^2}{3\alpha^3} - \frac{2g^3}{15\alpha^5} + \frac{17g^4}{315\alpha^7} - \ldots\ldots\ldots \tag{12}$$

We find that expression (12) is exactly same as obtained by the usual Born series method for the "θ" regularization of potential (10). Expression (12) is now summed up to yield:

$$A(\alpha) = -g^{\frac{1}{2}} \tanh\left(\frac{g^{\frac{1}{2}}}{\alpha}\right), \tag{13}$$

which in the limit $\alpha \to 0$ gives $-g^{\frac{1}{2}}$, an exact expression for the scattering length for the potential (10). Hence it is established that peratization is successful in this case.

It may be of interest to note here that for the " + " regularization [$V(r)$ replaced by $V(r+\alpha)$], the technique discussed in section 2.1, leads to the final result:

$$A(\alpha) = -\frac{g}{3\alpha} + \frac{g^2}{45\alpha^3} - \frac{2g^3}{945\alpha^5} + \ldots\ldots\ldots\ldots\ldots\ldots,$$

or $$A(\alpha) = -g^{\frac{1}{2}} \coth\left(\frac{g^{\frac{1}{2}}}{\alpha}\right) + \alpha, \tag{14}$$

and $$\lim_{\alpha\to 0} A(\alpha) = -g^{\frac{1}{2}} = A \tag{15}$$

where A is the exact scattering length.

Equation (15) shows that even for the " + " regularization of potential (10), the technique developed in section 2.1 is successful.

2.2 Application of the Theory to a Logarithmic Singular Potential

In order to see whether the theory developed in 2.1 works for other potentials as well, we apply it to a physically more realistic potential which is logarithmically singular in nature and is given by

$$V(r) = g\frac{\ln^2 r}{r^4} \tag{16}$$

It may be noted that the above potential has been discussed by Aly *et al* [28] in connection with the peratization technique. It can easily be shown that the scattering amplitude for the potential exists at zero energy. The peratized scattering length for "θ" regularization of potential (16) can be written directly through the application of the theorem given by Spector [29] as:

$$A(\alpha) = -g^{\frac{1}{2}}(\ln\alpha)\tanh\left(g^{\frac{1}{2}}\frac{\log\alpha}{\alpha}\right). \tag{17}$$

Thus different amplitudes $A_n(\alpha)$ obtained with the help of (7) are:

$$A_1(\alpha) = -\frac{g}{\alpha}\{\ln^2\alpha + 2\ln\alpha + 2\} \tag{18a}$$

$$A_2(\alpha) = -\frac{g}{\alpha^3}\left\{\frac{\ln^4\alpha}{3} + \frac{7\ln^3\alpha}{9} + \frac{17\ln^2\alpha}{18} + \frac{17\ln\alpha}{27} + \frac{17}{81}\right\} \tag{18b}$$

$$A_3(\alpha) = -\frac{g}{\alpha^5}\left\{\frac{2\ln^6\alpha}{15} + \ldots\ldots\ldots\right\} \tag{18c}$$

and so on.

We find that the above expressions are identical to those obtained by Aly *et al* [28] by the usual Born series method which further shows the correctness of the technique developed by us.

Now to apply the limiting process ($\alpha \to 0$) of the peratization, the terms of highest singularity in α in each order in *g*, are retained and the summation is made after which the limit $\alpha \to 0$ is applied. Retaining most singular terms in α in each order in *g* is called first-order peratization.

Thus isolating the leading singularities in α from (18), the following form for $A(\alpha)$ is obtained:

$$A(\alpha) = -g^{\frac{1}{2}}(\ln\alpha)\left\{g^{\frac{1}{2}}\frac{\ln\alpha}{\alpha} - \frac{g^{\frac{3}{2}}}{3}\frac{\ln^3\alpha}{\alpha^3} + \frac{2g^{\frac{5}{2}}}{15}\frac{\ln^5\alpha}{\alpha^5} - \ldots\ldots\ldots\right\}$$

or

$$A(\alpha) \approx -g^{\frac{1}{2}}(\ln\alpha)\tanh\left(g^{\frac{1}{2}}\frac{\ln\alpha}{\alpha}\right). \tag{19}$$

In the limit $\alpha \to 0$, this gives

$$A \sim -g^{\frac{1}{2}}(\ln\alpha), \tag{20}$$

which is not defined in the limit $\alpha \to 0$, thereby showing that the first order peratization does not succeed in this case.

Thus it can be conjectured that the agreement of our result (13), (15) and (18) with the results obtained by the usual Born series method shows the correctness of our method. The relative merit of the technique developed in this section is that the calculations involved for the scattering length are comparatively easier than other methods. It may also be noted that the potentials considered in sections 2.2 and 2.3 as examples contain only one term, hence the summation (1) of the power series in g is taken over from $n = 1$ to ∞. However, for the potentials which consist of two or more terms with one term independent of the coupling constant g {e.g. $V(r) = \frac{\left(g e^{\frac{2}{r}} + g'\right)}{r^4}$} the summation (1) {as well as (4)} should be taken over from $n = 0$ to ∞. In such cases it is found that except equation (7a) which is a first order non-linear differential equation, other equations are simple linear differential equations which can be solved by the integrating factor method.

3. Approach 2: (An Iterative Approach)

As stated in section 1, the usual procedure of obtaining series (1) so far has been through the Born series. After discussing a new approach to this procedure in section 2, we now discuss yet another approach which may be called an iterative approach. First we formulate this approach in the following section 3.1, and then apply it to different potentials in sections 3.2 and 3.3. Finally in section 4, we establish a relationship between the two approaches developed in this paper.

3.1 Formulation of the Method

As done before, we assume that series (1) can be obtained from (4) in the limit $r \to \infty$, yielding series (6).

In the following, we obtain series (4) through the iteration approach applied to the standard equation for scattering length which is given as:

$$a'(r) = -gV(r)[r + a(r)]^2 . \tag{21}$$

For this we set

$$a'_{m+1}(r) = -gV(r)[r + a_m(r)]^2 , \tag{22}$$

with $$a_0(r) = 0 . \tag{23}$$

Here $a_m(r)$ is the mth iteration of scattering length. Equation (22) for regularized potential $gV(r,\alpha)$ changes to

$$a'_{m+1}(r,\alpha) = -gV(r,\alpha)[r + a_m(r,\alpha)]^2 \tag{24}$$

with $$a_0(r,\alpha) = 0 \tag{25}$$

On repetition of the above process, a number of terms are obtained for the expansion $a_m(r,\alpha)$ which ultimately yield series (4). However, it should be noted that the *mth* order iteration $a_m(r,\alpha)$ contains terms of the order higher than in g which never need to be computed. As done in the previous section, the limits of integration for "θ" regularization here are from $\gamma = \alpha$ to γ and for "+"regularization from $\gamma = 0$ to γ in (24).

3.2 Application to Inverse Fourth Power Potential

We now apply the theory developed in section 3.1 to the potential

$$V(r) = \frac{g}{r^4}. \tag{26}$$

This potential, attractive or repulsive, has been studied most extensively [16, 26-27]. Dombey and Jones have also studied the inverse fourth power potential for its Regge behaviour [17].

Now for "θ" regularization of this potential (24) and (25) yield,

$$a_1(r,\alpha) = g\left\{\frac{1}{r} - \frac{1}{\alpha}\right\}. \tag{27a}$$

$$a_2(r,\alpha) = a_1(r,\alpha) + 2g^2\left\{\frac{1}{3r^3} - \frac{1}{2\alpha r^2} + \frac{1}{6\alpha^3}\right\} + 0(g^3), \tag{27b}$$

m

$$a_3(r,\alpha) = a_2(r,\alpha) + g^3\left\{\frac{7}{15r^5} - \frac{1}{\alpha r^4} + \frac{1}{3\alpha^2 r^3} + \frac{1}{3\alpha^3 r^2} - \frac{2}{15\alpha^5}\right\} + 0(g^4) \tag{27c}$$

$$a_4(r,\alpha) = a_3(r,\alpha) + 2g^4\left\{\begin{array}{l}\dfrac{17}{105r^7} - \dfrac{4}{9\alpha r^6} + \dfrac{4}{15\alpha^2 r^5} + \dfrac{1}{6\alpha^3 r^4} - \dfrac{1}{9\alpha^4 r^3} \\ -\dfrac{1}{15\alpha^5 r^2} + \dfrac{17}{630\alpha^7}\end{array}\right\} + 0(g^5) \tag{27d}$$

and so on. On increasing the order of iteration, series (4) is automatically obtained.

In the limit $r \to \infty$, this series leads to

$$\alpha(\alpha) = \lim_{r\to\infty} \alpha(r,\alpha) = -\frac{g}{\alpha} + \frac{g^2}{3\alpha^3} - \frac{2g^3}{15\alpha^5} + \frac{17g^4}{315\alpha^7} - \ldots\ldots\ldots\ldots$$

or $$a(\alpha) = -g^{\frac{1}{2}} \tanh\left(\frac{g^{\frac{1}{2}}}{\alpha}\right), \tag{28}$$

which in the limit $\alpha \to 0$ gives $-g^{\frac{1}{2}}$, which is an exact expression for the scattering length for potential (26). Thus it can be stated that peratization is successful for this case.

Similarly, for "+" regularization of potential (26), the application of the theory developed in section (3.1) give the following final expression for $\alpha(\alpha)$:

$$a(\alpha)=\lim_{r\to\infty} a(r,\alpha)=-\frac{g}{3\alpha}+\frac{g^2}{45\alpha^3}-\frac{2g^3}{945\alpha^5}+.........\qquad(29)$$

or
$$a(\alpha)=-g^{\frac{1}{2}}\cot h\left(\frac{g^{\frac{1}{2}}}{\alpha}\right)+\alpha\ ,\qquad(30)$$

and
$$\lim_{\alpha\to 0} a(\alpha)=-g^{\frac{1}{2}}.\qquad(31)$$

Thus we observe that even for the "+" regularization, iteration approach to peratization succeeds.

3.3 Application to Logarithmic Singular Potential

We next apply the theory to the following singular potential which is logarithmic in nature, it should be noted that this potential has been used by Aly *et al* [28] for studying peratization technique

$$V(r)=g\frac{\ln^2 r}{r^4}.\qquad(32)$$

For "θ" regularization of this potential, different iterations $a_1(r,\alpha)$, $a_2(r,\alpha)$, ..., etc are obtained with the help of (24) and (25). After applying the limit $r\to\infty$ to $a(r,\alpha)$ and isolating the leading singularities in α, in each order in g one gets for the first order peratization

$$a(\alpha)=-g^{\frac{1}{2}}(\ln\alpha)\left\{g^{\frac{1}{2}}\frac{\ln\alpha}{\alpha}-\frac{g^{\frac{3}{2}}}{3}\frac{\ln^3\alpha}{\alpha^3}+\frac{g^{\frac{5}{2}}}{15}\frac{\ln^5\alpha}{\alpha^5}-..........\right\}$$

or
$$a(\alpha)\approx -g^{\frac{1}{2}}(\ln\alpha)\tanh\left(g^{\frac{1}{2}}\frac{\ln\alpha}{\alpha}\right).\qquad(33)$$

It may be of interest to note that the result given above can also be obtained directly through the theorem given by Spector [28].

Now $\alpha(\alpha)$ in the limit $\alpha\to 0$ yields

$$a=-g^{\frac{1}{2}}(\ln\alpha)\cdot\qquad(34)$$

This result is identical to the one obtained by Aly *et al* [28].

4. CONCLUDING REMARKS

Following the notations for *A*'s from equations (1), (4) and (5), a relationship between iterations $a_n(r,\alpha)$ and *A*'s can be established. In fact, we observe that

$$a_1(r,\alpha)=gA_1(r,\alpha),\qquad(35a)$$

$$a_2(r,\alpha) = gA_1(r,\alpha) + g^2 A_2(r,\alpha), \tag{35b}$$

$$a_3(r,\alpha) = gA_1(r,\alpha) + g^2 A_2(r,\alpha) + g^3 A_3(r,\alpha), \tag{35c}$$

and so on, where $A_1(r,\alpha), A_2(r,\alpha), A_3(r,\alpha),$......, etc have been defined in section 2 of this paper. Thus it can easily be seen that

$$a(r,\alpha) \equiv A(r,\alpha), \tag{36}$$

and hence, in the limit $r \to \infty$

$$a(\alpha) \equiv A(\alpha). \tag{37}$$

which explains the similarity between the results obtained in the two approaches. Further it has also been shown that the results obtained by the two approaches for peratization completely agree with the usual peratization technique. This proves the correctness of our two formulations.

REFERENCES

[1] Khuri N.N., Pais A., Singular Potential and Peratization I, *J. Math. Phys.*, 1970; **36:** 590

[2] Pais A., Wu T.T., Singular Potential and Peratization II, *J. Math. Phys.,*1964;**5:** 799.

[3] Wu T.T., Scattering by the Singular Potential -g r $^{-4}$ln r, *Phys. Rev.,* 1964; **B136:** 1176.

[4] Tiktopoulas G. and Trieman S.B., Weak Coupling limit for strongly singular potentials, *Phys. Rev.,* 1964; **B134:**844.

[5] Paliov A., and Rosendroff S., Exponentially small scattering amplitude in high-energy potential scattering, *J. Math. Phys.,* 1967; **8:**1829.

[6] Calogero F., Exponentially singular potentials and Peratization, *Phys. Rev.,* 1964; **B 135:** 693.

[7] Calogero F., *Variable phase approach to potential scattering,* Academic press, 1967; **69** .

[8] Spector R.M., High energy scattering cross-section for singular potentials at large angles, *J. Math. Phys.,* 1969; **10:** 503.

[9] Stanciu G.N., A peratization method for low energy scattering from singular potentials, *Nuovo. Cim.,* 1967; **A 50:** 293.

[10] Bethe H.A. and Kinoshita T., Behaviour of Regge Poles in a potential at large energy, *Phys. Rev.*, 1962; **128:** 1418.

[11] Vogt E., Wannier G., Scattering of ions by polarization forces, *Phys. Rev.,* 1954; **95:** 1190.

[12] Thaler R.M. and Biedenharn L.C., A note on the calculation of coulomb penetration and coulomb wave functions, *Nucl. Phys.*, 1957; **3:** 207.

[13] Gellmann M., Low F., Quantum Electrodynamics at Small Distances, *Phys. Rev.,* 1954**; 95:** 1300.

[14] Case K.M., Singular Potentials, *Phys. Rev.*, 1950; **80:** 797.

[15] Aly H.H., Müller H.J.W. and Schilcher K., Regge trajectories for even power potentials, *Nucl. Phys.*, 1967; **B 3**: 401.

[16] Challifour J. and Eden R.J., Regge Poles and branch cuts for potential scattering, *J. Math. Phys.*, 1963; **4:** 359.

[17] Dombey N. and Jones R., Analyticity properties of the scattering amplitude for singular potentials, *J. Math. Phys.,* 1968; **9:** 986.

[18] Gale W.A., Peratization of the logarithmically singular potential, *J. Math. Phys.,* 1966; **7:** 1271.

[19] Jabbur R.J., Asymptotic properties of the S-Matrix for singular potentials, *Phys. Rev.,* 1965; **B 138:** 1525.

[20] Fienberg G. and Pais A., A field theory of weak interactions, *Phys. Rev.,* 1963; **131:** 2724.

[21] Calogero F. and Cassandro M., Exponentially singular potentials and peratization, *Nuovo. Cim.,* 1965; **37**: 760.

[22] Cornille H., Limiting procedures for singular potentials, *Nuovo. Cim.,* 1966; **A 43**: 786.

[23] Gale W.A., Regularization and Peratization of singular potentials, *J. Math.Phys.,* 1967; **8**: 1050.

[24] Frank W.M. and Land D.J., Peratization of a class of exponentially singular potentials I, *J. Math. Phys.,* 1970 a; **11:** 2041.

[25] Frank W.M. and Land D.J., Peratization of a class of exponentially singular potentials II, *J. Math. Phys.,* 1970b; **11:** 2058.

[26] Spector R.M., High energy scattering cross-section for singular potentials, *J. Math. Phys.,* 1964; **5:** 1185.

[27] Aly H.H. and Müller H.J.W., Test for the N/D method in potential scattering, *J. Math. Phys.,* 1966; **7:**1.

[28] Aly H.H., Riazuddin and Zimmerman H: Singular potentials and peratization, *Phys. Rev.,* 1964; **B 136:**1174.

[29] Spector R.M., A theorem on peratization of singular potentials and other miscellanea, *J. Math. Phys.*, 1966; **7:**2103.