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Notes on the Primal Submodules

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ABSTRACT

In this note, we will make an intensive study of the notion of primal submodules of a module over a commutative ring with identity. First, a number of results concerning of primal submodules are given. Second, for a commutative ring R, the notion of primal Multiplication (resp. generalized primal multiplication module) over R is defined. Also, the relation among the families of weak multiplication modules, primal multiplication modules and generalized primal multiplication modules over a commutative ring is considered.

Keywords: primal, prime, multiplication.

1. INTRODUCTION

In this note, R will denote a commutative ring with nonzero identity, and all modules are unitary. Primal ideals in a commutative ring with identity have been introduced and studied by L. Fuchs in [1] (also see [2]). The concept primal submodules have been introduced by D. Dauns in [3]. Also, this class of submodules is studied extensively in [4]. The purpose of this paper is to study some further results and various properties of primal submodules of an R-module M. In Section 1, we first give an example (Example 2.1) in which we show that all submodules of the Z-module $E(p)$ are primal, where Z denotes the set of integers. This example also shows that a primal submodule need be neither prime, nor secondary and need not be an RD-submodule too. Then we give some conditions under which a primal submodule may be prime, secondary and an RD-submodule (See Theorems 2.2, 2.3 and 2.4). Submodules of primal submodules need not be primal (see Example 2.6); however, we show in Theorem 2.7 that under specific conditions this holds. Also we show in Theorem 2.8 that factor modules of primal submodules are again primal. Finally, we prove that if M is a finitely generated multiplication module over a Prüfer domain R, then a submodule of M is primal if and only if it is irreducible (see Theorem 2.13).

In section 3, we first introduce that the notion primal multiplication module over a commutative ring. Various properties of primal multiplication modules are considered. For example, every primal multiplication module is weak multiplication; however, a week multiplication need not be primal multiplication module (see Example 3.3). We show that every torsion-free primal multiplication module over an integral domain is indecomposable (Theorem 3.9). Also, we show that primal multiplication modules are really only of interest in indecomposable rings (see Theorem 3.13). Finally, we introduce that the notion of generalized primal multiplication modules over a commutative ring and we give some properties of such modules (see Theorems 3.16 and 3.17).

Now we define the concepts that we will use. Let R be a commutative ring, M an R-module and N an R-submodule of M. An element $r \in R$ is called prime to N if $rm \in N$ ($m \in M$) implies that $m \in N$. In this case, $(N :_M r) = \{m \in M \mid rm \in N\} = N$. Denote by $S(N)$ the set of all elements of R that are not prime to N. A proper submodule N of M is said to be primal if $S(N)$ forms an ideal of R; this ideal is called the adjoint ideal P of N. In this case we also say that N is a P-primal submodule of M (see [3, 4].

A proper submodule N of a module M over a ring R is said to be prime (P-prime) if $rm \in N$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r \in (N :_R M) = P$. The set of all prime submodules in an R-module M is denoted $Spec(M)$. Recall that if R is an integral domain with the quotient field K, the rank of an R-module M ($rank(M)$) is defined to be the maximal number of elements of M linearly independent over R. We have $rank\ M$ = the dimension of the vector space KM over K. An R-module M is called a weak multiplication module if $Spec(M) = \phi$ or for every prime submodule N of M we have $N=IM$, where I is an ideal of R [5]. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N :_R M)$.

An R-module $M \neq 0$ is called a secondary module (resp. second module) provided that for every element $r \in R$, the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent (resp. either surjective or zero). This implies that $nilrad(M) = P$ (resp. $Ann_R(M) = P'$) is a prime ideal of R, and M is said to be P-secondary (resp. P'-second) [6]. Let N be an R-submodule of M. Then N is called relatively divisible submodule (or an RD-submodule) if $N = N \cap rM$ for all $r \in R$.

2. PRIMAL SUBMODULES

In this section we list some basic properties concerning primal submodules. First, consider the following example:

Example 2.1 Let p be a fixed prime integer and $N_0 = Z^+ \cup \{0\}$. Then

$$E(p) = \{\alpha \in Q/Z \mid \alpha = r/p^n + Z \quad for \quad some \quad r \in Z \quad and \quad n \in N_0\}$$

is a non-zero submodule of the Z-module Q/Z. For each $t \in N_0$, set

$$G_t = \{\alpha \in Q/Z \mid \alpha = r/p^t + Z \quad for \quad some \quad r \in Z\}$$

For every $n \in N_0$, set $\alpha_n = 1/p^n + Z$. Then by [7, Example 7.10], $pG_{t+1} = G_t$, G_t is a cyclic Z-module with $G_t = Z\alpha_t$ such that $P\alpha_{t+1} = \alpha t$, every non-zero proper submodule H of $E(p)$ is of the form $H = G_n$ for some $n \in N_0$ and $E(p)$ is an Artinian Z-module (it is not Noetherian) with a strictly increasing sequence of submodules:

$$G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_n \subset G_{n+1} \subset \ldots$$

(1) By [3, Example 1.14], G_0 is a primal submodule of $E(p)$. We claim that, for every $t \in N_0$, G_t is a pZ-primal submodule of $E(p)$. If $pk \in pZ$, we have $pk\alpha_{t+1} = k\alpha_t \in G_t$ with $\alpha_{t+1} \notin G_t$. So pk is not prime to G_t. Now assume that $n \in Z$ is not prime to G_t. Then there exists a positive integer $k>t$ with $0 \neq a_k = r/p^k + Z \in E(p) \setminus G_t$ such that $na_k \in G_t$. So $n(r/p^k + Z) = r'/p^t + Z$ for some $r' \in Z$. We may assume that r and p are coprime. So there exists $m \in Z$ with $rn - r'p^{k-t} = p^k m$. It follows that $p \mid rn$ and hence $p \mid n$. Thus, pZ consists exactly of the set of elements of Z which are not prime to G_t. Hence G_t is a pZ-primal submodule of $E(p)$.

(2) Since $pG_{t+1} = G_t \neq G_{t+1} \cap pE(p) = G_{t+1}$ (Since $E(p)$ is divisible), we get G_{t+1} is not an RD-submodule of $E(p)$; hence by **(1)**, a primal submodule need not be an RD-submodule.

(3) By [8, p. 3745], the set of prime submodules of $E(p)$ is empty. Hence by **(1)**, a primal submodule need not be a prime submodule.

(4) Set $M=E(2)$. Then M is an indecomposable 0-secondary Z-module (see [9]), but G_2 is not secondary (since $3G_2 \neq G_2$ and for every positive integer n, $3^n G_2 \neq 0$), hence a primal submodule need not be a secondary submodule. □

Now consider the following results:

Theorem 2.2 Let R be a commutative ring, M an R-module and N a P-primal submodule of M. Then the following hold:

(i) If M is P-secondary, then N is P-secondary.

(ii) If M is P-second, then N is an RD-submodule of M.

(iii) If K is a P-secondary submodule of M, then $N \cap K$ is P-secondary.

Proof. **(i)** Let $r \in R$. If $r \in P$, then there exists a positive integer t such that $r^t N \subseteq r^t M = o$. If $r \notin P$, then $rM = M$. It suffices to show that $N \subseteq rN$. Let $n \in N$. Then there exists $m \in M$ such that $n=rm$. If $m \notin N$, then r is not prime to N; hence $r \in P$ which is a contradiction. Therefore, $n \in rN$, and so we have $rN=N$. Thus N is secondary.

(ii) Let $r \in R$. If $r \in P$, then $rN \subseteq rM = 0$, so $rN = 0 = N \cap rM$. So suppose that $r \notin P$. Since the inclusion $rn \subseteq N \cap rM$ is trivial, we will prove the reverse inclusion. Let $n = rm \in N \cap rM$ for some $m \in M$. Then r prime to N gives $m \in N$; so $n \in rN$, and so we have equality.

(iii) Let $r \in R$. If $r \in P$, there exists a positive integer t such that $r^t(N \cap K) \subseteq r^t K = 0$. Suppose that $r \notin P$; we show that $r(N \cap K) = N \cap K$. It is enough to show that $N \cap K \subseteq r(N \cap K)$. If $x \in N \cap K \subseteq K = rK$, then there exists $c \in K$ with $x=rc$. Since $x \in N$ and r is prime to N, we must have $c \in N$; hence $c \in N \cap K$. So $x = rc \in r(N \cap K)$, as required. □

Let M be a module over a commutative ring R. Then by [3, Proposition B], every prime submodule of M is primal. Furthermore, by Example 2.1, a primal submodule of M need not be prime; however, we have the following Theorem:

Theorem 2.3 Let N be a P-primal submodule of an R-module M. Then N is a P-prime submodule of M if and only if $P \subseteq (N :_R M)$.

Proof. It suffices to show that if $P \subseteq (N :_R M)$, then N is a P-prime submodule of M. By [4, Lemma 2.2], we must have $P = (N :_R M)$. Suppose that $rm \in N$ for some $r \in R$ and $m \in M$. If $m \notin N$, then r is not prime to N; hence $r \in P = (N :_R M)$, as needed. □

By Example 2.1, a primal submodule of M need not be a RD-submodule of M; however, we have the following theorem:

Theorem 2.4 Let R be an integral domain and let M be an R-module. Then every 0-primal submodule of M is an RD-submodule.

Proof. Assume that N is a 0-primal submodule of M and let $r \in R$. Clearly, $rN \subseteq N \cap rM$. For the reverse inclusion, assume that $n = rm \in N \cap rM$ for some $m \in M$. If $m \in N$, then $n \in rN$; otherwise, $m \in (N :_M r) \setminus N$. Thus r is not prime to N and hence $r = 0$. Therefore, $n = 0 \in rN$, and so we have equality. □

Theorem 2.5 Let R be a commutative ring and M a finitely generated R-module. Then every P-primal submodule of M may be embedded in a P-prime submodule of M.

Proof. Let N be a P-primal submodule of M. It follows from [4, Lemma 2.2] and [10, Theorem 3.3] that there exists a prime submodule K of M such that $N \subseteq K$ and $(K :_R M) = P$. □

Example 2.6 Consider Z, the set of integers, as a Z-module. We claim that $6Z$ is a $3Z$-primal submodule of $2Z$. Clearly every element of $3Z$ is not prime to $6Z$. Now assume that

$n \in Z$ is not prime to *6Z*. There exists $a = 2k \in 2Z \setminus 6Z$ such that $an \in 6Z$. So 3 divides *kn*. As 3 doesn't divide *k*, we must have $n \in 3Z$. We have already shown that *3Z* consists exactly of those elements of *Z* that are not prime to *6Z*. Therefore, *6Z* is a *3Z*-primal submodule of *2Z*. Moreover, *2Z* is a *2Z*-primal submodule of *Z* (since it is a prime submodule). However, *6Z* is not a primal submodule of *Z* since 2 and 3 are not prime to *6Z* but 3-2=1 is prime to *6Z*. This example shows that a submodule of a primal submodule need not be again primal. □

Now consider the following theorems:

Theorem 2.7 Let *R* be a commutative ring, *M* an *R*-module and *N* and *K, R*-submodules of *M* with $K \subseteq N$. Then the following holds:

(i) If *K* is a *P*-primal submodule of *N* and $S(N) \subseteq P$, then *K* is a *P*-primal submodule of *M*.

(ii) Let *(R, P)* be a local ring. If *K* is a *P*-primal submodule of *N* and *N* is a primal submodule of *M*, then *K* is a primal submodule of *M*.

(iii) Let *K* be a *P*-primal submodule of *N* and *N* a *Q*-primal submodule of *M*. If *Q* is a nil ideal of R and *K* is a primal submodule of *M*, then either $P \subseteq Q$ or $Q \subseteq P$.

Proof. (i) If $r \in R$ is not prime to *K*, then there exists $m \in M \setminus K$ with $rm \in K$. If $m \in N$, then $r \in P$. So suppose that $m \notin N$. Therefore, $r \in S(N) \subseteq P$. Now let $r \in P$. There exists $n \in N \setminus K$ such that $rn \in K$. Thus $n \in M \setminus K$ gives *r* is not prime to *K*. Thus *K* is a *P*-primal submodule of *M*.

(ii) This follows from (i).

(iii) Let *K* be a P'-primal submodule of *M*. It suffices to show that $P' = P \cup Q$. Since the inclusion $P' \subseteq P \cup Q$ is clear, we will prove the reverse inclusion. Let $r \in P \cup Q$. If $r \in P$, then there is an element $n \in N \setminus K$ with $rn \in K$, so $n \in M \setminus K$ gives $r \in P'$. So suppose that $r \in Q \setminus P$. Then there exists $m \in M \setminus N$ such that $rm \in N$. If $rm \notin K$, then *Q* nil ideal gives there is a non-negative integer *t* with $r^{t+1} = 0$; hence $r^t(rm) = 0 \in K$ gives $r^t \in P$ which is a contradiction. So $rm \in K$, and hence $r \in P'$, as required. □

Theorem 2.8 Let *R* be a commutative ring, *M* an *R*-module and *N, K R*-submodules of *M* with $K \subseteq N$. Then *N* is a *P*-primal *R*-submodule of *M* if and only if *N/K* is a *P*-primal *R*-submodule of *M/K*.

Proof. Assume that *N* is a *P*-primal submodule of *M* and let $r \in P$. Then there exists $m \in M \setminus N$ such that $rm \in N$; hence $r(m+K) \in N/K$ with $m + K \notin N/K$. This implies that *r* is not prime to *N/K*. Now assume that $r \in R$ is not prime to *N/K*. Then there exists $m + K \notin N/K$ with $r(m+K) \in N/K$. Hence $m \notin N$ and $rm \in N$ which implies that *r* is not prime to *N*; hence *P=S(N/K)*. Thus *N/K* is a *P*-primal submodule of *M/K*. The reverse implication is similar and we omit it. □

In [4], a number of results concerning primal submodules in the module of fractions are given. Now we give further information about primal submodules in the module of fractions.

Theorem 2.9 Assume that R is a commutative ring and let N be a non-zero submodule of an R-module M such that $(N :_R m) = P$ is a prime ideal of R for some $m \in M \setminus N$. Then the submodule

$$N_P \cap M = \{m \in M \mid m/1 \in N_P\}$$

is a P-primal submodule of M.

Proof. Set $B = N_P \cap M$. First, we show that $(B :_R m) = P$. On one hand, we have $Pm \subseteq N \subseteq B$, so $P \subseteq (B :_R m)$. On the other hand, it is enough to show that if $r \notin P$, then $r \notin (B :_R m)$. Suppose that $r \in (B :_R m)$. Then $(rm)/1 \in N_P$, so $(rm)/1 = x/s$ for some $x \in N$ and $s \in R \setminus P$. Therefore, there exists $t \in R \setminus P$ such that $rtsm = tx \in N$, so $rts \in P$, a contradiction. Thus $(B :_R m) = P$; hence as $m \notin B$, no element contained in P is prime to B. Next, we will prove that every $b \notin P$ is prime to B. Clearly, $B \subseteq (B :_M b)$. For the reverse inclusion, assume that $y \in (B :_M b)$, so $(by)/1 = n/c \in N_P$ for some $n \in N$ and $c \in R \setminus P$. It follows that $btcy = tn \in N$ for some $t \in R \setminus P$. As $y/1 = (tn)/(btc) \in N_P$, we get $y \in B$, as required. □

Theorem 2.10 Let R be a commutative ring, M an R-module and N an R-submodule of M. If $(N :_R M)$ is a P-primal ideal of R, then $N = N_P \cap M$ if and only if N is a P-primal submodule of M.

Proof. Assume that $N = N_P \cap M$ and let $a \in P$. Since $(N :_R M)$ is a P-primal ideal of R, we must have $((N :_R M) :_R a) \supset (N :_R M)$; hence there exists $r \in ((N :_R M) :_R a) \setminus (N :_R M)$. Thus $raM \subseteq N$ and $rM \not\subset N$, so $rm \notin N$ for some $m \in M$. But $a(rm) = ram \in raM \subseteq N$; hence $rm \in (N :_M a) \setminus N$. Therefore, a is not prime to N. Now assume that $b \in R$ is not prime to N. Then $(N :_M b) \supset N$; hence there exists $m \in (N :_M b) \setminus N$. If $b \notin P$, then, $m/1 = (bm)/b \in N_P$, so $n \in N_P \cap M = N$ which is a contradiction; hence $b \in P$. Thus P is exactly the set of elements of R which are not prime to N. So N is a P-primal Submodule of M. The converse follows from [4, Proposition 2.8]. □

Let R be a commutative ring, M an R-module and S a multiplicatively closed subset of R. For every submodule N of M, let

$$N_S = \{m \in M \mid sm \in N \quad for \quad some \quad s \in S\}.$$

It is clear that N_S is a submodule of M containing N. Also if $(N :_R M) \cap S \neq \phi$, then $N_S = M$. Let P be a prime ideal of R and set $S_P = R \setminus P$. Then $m \in N_{S_P}$ if and only if $(N :_R m) \not\subset P$. Furthermore $N_{S_P} = N_P \cap M$ where N_p is the localization of N at P.

Theorem 2.11 Let N be a finitely generated submodule of an R-module M such that $(N :_R M) \subseteq P$ for some $P \in Supp(N)$. Then $(PN)_{S_P}$ is a P-primal submodule of M.

Proof. By [4, Theorem 2.3], it suffices to show that $P/I = Zdv_{R/I}(M/K_{S_P})$, where $K = PN$ and $I = (K_{S_P} :_R M)$. Let $r+I$ is an element of $Zdv_{R/I}(M/K_{S_P})$. Then there exists a non-zero element $m + K_{S_P}$ in M/K_{S_P} such that $(r+I)(m+K_{S_P}) = 0$; hence $srm \in K$ for some $s \in S_P$. If $r \notin P$, then $m \in K_{S_P}$ which is a contradiction. Thus we must have $Zdv_{R/I}(M/K_{S_P}) \subseteq P/I$. For the other containment, assume that $r + I \in P/I$. As $N_P \neq 0$, Nakayama's Lemma gives $K_P \subset N_P$. So there exists $n/s \in N_P \setminus K_P$ with $n/1 = (sn)/s \in N_P$; hence $n \in N_P \cap M = N_{S_P}$. Moreover, $n \notin K_{S_P}$ (otherwise, there exists $t \in S_P$ with $tn \in K$, so $n/s = (tn)/(ts) \in K_P$ which is a contradiction). Therefore, $K_{S_P} \subset N_{S_P}$. Also, $N_{S_P} \subseteq (K_{S_P} :_M r)$ (for if $n \in N_{S_P}$, then $sn \in N$ for some $s \in S_P$, so $s(rn) = r(sn) \in PN = K$; hence $rn \in K_{S_P}$). Therefore, $K_{S_P} \subset (K_{S_P} :_M r)$. So there exists $m \in (K_{S_P} :_M r) \setminus K_{S_P}$. Hence $(r+I)(m+K_{S_P}) = rm + K_{S_P} = 0$ which implies that $r + I \in Zdv_{R/I}(M/K_{S_P})$, as needed. □

In the end of this section we give some information about primal submodules of a multiplication module. Set $R = Z$, and let p be a fixed prime integer. If we adapt the proof of the well- known fact that $E(p)$ is divisible, then we obtain every non-zero proper submodule of the Z-module $E(p)$ is a multiplication module, but the Z-module $E(p)$ itself is not (see Example 3.15).

Proposition 2.12 Let R be a commutative ring, M a finitely generated multiplication R-module, $N=IM$ an R-submodule of M and $a \in R$. Then the following hold:

(i) $(I :_R a) = I$ if and only if $(N :_M a) = N$.

(ii) a is not prime to I if and only if a is not prime to N.

(iii) I is a P-primal ideal of R if and only if N is a P-primal submodule of M.

Proof. (i) Set $B = (N :_M a)$. By assumption, we have $B = (B :_R M)M$ and

$$(I :_R a)M \subseteq B = (B :_R M)M \subseteq (N :_R aM)M = (I :_R a)M .$$

Thus $(I :_R a)M = (N :_M a)$. If $(I :_R a) = I$, then $(I :_R a)M = N = (N :_M a)$. If $N = (N :_M a)$, then $IM = (I :_R a)M$, so by [11, p. 231 Corollary] we have $I + (0 :_R M) = (I :_R a) + (0 :_R M)$; so $(I :_R a) = I$ since $(0 :_R M) \subseteq I \subseteq (I :_R a)$.

(ii) This follows from (i).

(iii) Suppose first that N is a primal submodule of M and let P be the set of elements of R that are not prime to I. We show that P is an ideal of R. Let $a,b \in P$. Then $(I :_R a) \neq I$ and $(I :_R b) \neq I$. If $(I :_R a-b) = I$, then $N = (N :_M a-b)$, which is a contradiction by (ii). Thus, $a - b \in P$. Similarly, if $a \in P$ and $r \in R$, then $(I :_R ra) \neq I$. Therefore, P is an ideal of R. The other implication is similar, and we omit it. □

In general, a primal submodule need not be necessarily irreducible; however, we have the following theorem:

Theorem 2.13 Let R be a Prüfer domain, M a finitely generated multiplication R-module and N an R-submodule of M. Then N is irreducible if and only if N is primal.

Proof. By [3, Proposition 1.9], it is enough to show that if N is primal, then N is irreducible. There is an ideal I of R such that $N=IM$. Since N is primal, it then follows from Proposition 2.12 that I is a primal ideal of R; hence I is irreducible by [2, Lemma 2.6]. Now the assertion follows from [12, Proposition 3.2]. □

3. PRIMAL MULTIPLICATION MODULES

In this section, our starting point is the following definition:

Definition 3.1 Let R be a commutative ring. An R-module M is called a primal multiplication module if either M has no primal submodules or for every primal submodule N of M, $N=IM$ for some ideal I of R.

It is easy to show that if M is a primal multiplication module, then $N=(N:_R M)M$ for every primal submodule N of M.

Proposition 3.2 Let M be a divisible R-module. Then M is primal multiplication if and only if it has no non-zero primal submodules.

Proof. The sufficiency is clear. Conversely, assume that M is primal multiplication and let N be a non-zero primal submodule of M. Then $N=IM=M$ for some ideal I of R, which is a contradiction, as required. □

Example 3.3 Let p, $E(p)$ and be as described in Example 2.1. Then:

(1) By [8, P.3745], $Spec(E(p))=\phi$. Hence $E(p)$ is a weak multiplication Z-module.

(2) For every $t\in N_0$, Example 2.1 and Propositin 3.2 gives $E(p)$ is not a primal multiplication module. Thus, a weak multiplication module need not be primal multiplication. □

If R is a ring (not necessarily an integral domain) and M is an R-module, the subset $T(M)$ of M is defined by

$$T(M)=\{m\in M \mid rm=0 \quad for \quad some \quad 0\neq r\in R\}.$$

It is clear that if R is an integral domain, then $T(M)$ is a submodule of M. In this case, M is called torsion-free if $T(M)=0$, and it is called torsion if $M=T(M)$. We recall that, by [3, Proposition 1.9], over a commutative ring R, every prime submodule is primal; hence every primal multiplication module is weak multiplication. By using this fact and the results in [5], we have the following Remarks:

Remark 3.4 Let M be a primal multiplication module over an integral domain R. Then:

(1) If M is a non-zero torsion-free module, then $rank(M)=1$.
(2) If M is a torsion module, then $rank(M)=0$.
(3) M is either torsion or torsion-free. □

Remark 3.5 Let M be a primal multiplication module over a commutative ring R. Then:

(1) If M is finitely generated, then M is multiplication.
(2) If R is an Artinian ring, then M is cyclic. □

Theorem 3.6 Every homomorphic image of a primal multiplication module is a primal multiplication.

Proof. Assume that M is a primal multiplication R-module and let $f: M \to M'$ be an R-epimorphism. If N' is a primal submodule of M', then Theorem 2.8 gives $f^{-1}(N')$ is a primal submodule of M, so there exists an ideal I of R with $f^{-1}(N') = IM$; hence $N' = IM'$, and the proof is complete. □

Lemma 3.7 Let M be a primal multiplication module over an integral domain R. If N is a 0-primal submodule of M, then $N=0$.

Proof. Since M is a primal multiplication module, we must have $N = (N :_R M)M \subseteq 0M = 0$ by [4, Lemma 2.2(i)], as required. □

Proposition 3.8 Let M be a module over an integral domain R. If $M = N \oplus K$ for some proper R-submodules N and K with K torsion-free, then N is a 0-primal submodule of M.

Proof. Suppose that $r \in R$ is not prime to N; we show that $r=0$. Then there exists $m \in M - N$ with $rm \in N$. We can write $m = n + k$ for some $n \in N$ and $k \in K$, so $rk \in N \cap K = 0$. If $r \neq 0$, then $k = 0$ which is a contradiction. Thus, N is a 0-primal submodule of M. □

Theorem 3.9 Every torsion-free primal multiplication module over an integral domain is indecomposable.

Proof. This follows from Lemma 3.7 and Proposition 3.8. □

Theorem 3.10 Let R be a discrete valuation ring. Then the following hold:

(i) Every non-zero primal multiplication module over R is indecomposable.
(ii) Every primal multiplication module over R is multiplication.

Proof. (i) Assume that $P = pR$ is the unique maximal ideal of R and let $M = N \oplus K$ for some non-zero proper submodules N and K of M. By [4, Proposition 2.11], N and K are primal submodules of M. So there are integers s,t $(s < t)$ such that $N = p^s M$ and $K = p^t M$. In this case $0 = N \cap K = p^t M$ which is a contradiction. Thus M is indecomposable.

(ii) This follows from [4, Proposition 2.11]. □

We next show that primal multiplication modules are really only of interest in indecomposable rings. We first note two facts concerning the stability of primalness.

Theorem 3.11 Let M be a module over a commutative ring R. Then the following hold:

(i) If S is a multiplicatively closed subset of R and M is primal multiplication, then $S^{-1}M$ is a primal multiplication $S^{-1}R$-module.

(ii) If (R, P) is a local ring, then M is a primal multiplication R-module if and only if M_P is a primal multiplication R_P-module.

Proof. (i) Let B be a primal submodule of $S^{-1}M$. Then by [4, Proposition 2.8], $B \cap M$ is a primal R-submodule of M, so there exists an ideal I of R with $B \cap M = IM$; hence $B = S^{-1}(B \cap M) = S^{-1}(IM) = (S^{-1}I)(S^{-1}M)$. Thus $S^{-1}M$ is primal multiplication.

2) By (i), it suffices to show that if M_P is a primal multiplication R_m-module, then M is a primal multiplication R-module. Let N be a P'-primal submodule of M. Then $P' \subseteq P$ and [4, Proposition 2.7] gives N_P is a primal submodule of M_P; hence . $N_P = (N_P :_{R_P} M_P)M_P$ Hence, by [4, Theorem 2.6], $N_P = ((N :_R M)M)_P$. So $N = (N :_R M)M$, as required. □

Proposition 3.12 Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. Let M_i be an R_i-module and let $M = M_1 \times M_2$ be the R-module with action $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ where $r_i \in R_i$ and $m_i \in M_i$. Then an R-submodule N of M is primal if and only if either $N = N_1 \times M_2$ or $N = M_1 \times N_2$, where N_1 and N_2 are primal submodules of M_1 and M_2, respectively.

Proof. Assume that N is a P-primal submodule of M. It is well-known that $P = P_1 \times R_2$ or $P = R_1 \times P_2$ where P_i is a prime ideal of R_i, for $i=1, 2$. Suppose that $P = P_1 \times R_2$ and let $N = N_1 \times N_2$; we show that $N_2 = M_2$. If $m_2 \in M_2$, then $(1,0)(0, m_2) = (0,0) \in N$ and $(1,0) \notin P$ gives $(0, m_2)$ is prime to N; hence $m_2 \in N_2$. Thus $N_2 = M_2$. Since N is a P-primal submodule of M, we must have $(N :_R M) \subseteq P$ and $Zdv_{R/(N :_R M)}(M/N) = P/(N :_R M)$ by [4, Lemma 2.2 and Theorem 2.3]. Hence

$$(N_1 :_{R_1} M_1) \times R_2 = (N :_R M) \subseteq P = P_1 \times R_2$$

implies that $(N_1 :_{R_1} M_1) \subseteq P_1$. Also it is easy to check that

$$Zdv_{R_1/(N_1 :_{R_1} M_1)}(M_1 / N_1) = P_1 / (N_1 :_{R_1} M_1).$$

Therefore, N_1 is a P_1-primal submodule of M_1 by [4, Theorem 2.3]. The case where $P = R_1 \times P_2$ is similar.

Conversely, assume that N_1 is a P_1 primal submodule of M_1. By [4, Theorem 2.3], $(N_1 :_{R_1} M_1) \subseteq P_1$ and $Zdv_{R_1/(N_1:_{R_1}M_1)}(M_1/N_1) = P_1/(N_1 :_{R_1} M_1)$. Assume that $N = N_1 \times M_2$. Then, $(N :_R M) = (N_1 :_{R_1} M_1) \times R_2 \subseteq P_1 \times R_2$ and $Zdv_{R/(N:_R M)}(M/N) = P/(N :_R M)$. Hence, again by [4, Theorem 2.3], N is a primal submodule of M. If N_2 is a primal submodule of M_2, then by a similar argument we see that $M_1 \times N_2$ is a primal submodule of M. □

Theorem 3.13 Let $R = R_1 \times R_2$ where each R_i is a commutative ring with identity. Let M_i be an R_i-module and let $M = M_1 \times M_2$ be the R-module with the action $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$ where $r_i \in R_i$ and $m_i \in M_i$. Then M is a primal multiplication R-module if and only if M_1 and M_2 are primal multiplication R_i-modules, $i=1, 2$.

Proof. Assume that M is a primal multiplication R-module and let N_1 be a primal submodule of M_1. By Proposition 3.12, $N_1 \times M_2$ is a primal submodule of M; hence $N_1 \times M_2 = IM$ for some ideal $I = I_1 \times I_2$ of R, where I_i is an ideal of R_i, $i=1, 2$. Then $N_1 = I_1M_1$ gives M_1 is a primal multiplication R_1-module. Similarly, one can show that M_2 is a primal multiplication R_2-module. Conversely, assume that both M_1 and M_2 are primal multiplication. Let N be a primal submodule of M. Then, by Proposition 3.12, we can assume that $N = N_1 \times M_2$, where N_1 is a primal submodule of M_1. As M_1 is primal multiplication, there exists an ideal I_1 of R_1 such that $N_1 = I_1M_1$. Now $N = N_1 \times M_2 = (I_1M_1) \times M_2 = (I_1 \times R_2)(M_1 \times M_2)$. Thus, M is primal multiplication. □

Definition 3.14 An R-module M is called generalized primal multiplication if every proper submodule of M is primal multiplication.

Example 3.15

Let p, $E(p)$ and G_t be as described in Example 2.1. Pick $t \in N_0$ and fix it. It is easy to see that, for every $m < t$, G_m is a pZ-primal submodule of G_t. Moreover, for any $m<t$, $G_m = p^{t-m}G_t$. Hence G_t is primal multiplication submodule of $E(p)$. It follows that $E(p)$ is a generalized primal multiplication module. Furthermore, by Example 3.3, $E(p)$ is not primal multiplication. Therefore, a generalized primal multiplication module need not be primal multiplication. □

Theorem 3.16 If M is a generalized primal multiplication module over an integral domain R, then M is either torsion or torsion-free.

Proof. Suppose that M is not torsion-free; we show that M is torsion. Otherwise, there exists a proper submodule $N/T(M)$ of $M/T(M)$. We show that $T(M)$ is a primal submodule of N. Assume that $r \in R$ is not prime to $T(M)$ as a submodule of N. Then, $rn \in T(M)$ for some $n \notin T(M)$. Since $T(M)$ is a 0-prime submodule of M with $(T(M) :_R M) = 0$, we must have

$r=0$. Therefore, $T(M)$ is a 0-primal submodule of N. Then N primal multiplication gives there exists a nonzero ideal I of R with $T(M)=IN$. Assume that $0 \neq a \in I$ and let $n \in N$. Then $an \in T(M)$, so there is $0 \neq b \in R$ such that $ban=0$; hence $n \in T(N)$. It follows that $N = T(N) \subseteq T(M)$ which is a contradiction, as required. □

Theorem 3.17 Let R be a commutative ring, M a generalized primal multiplication R-module and N an R-submodule of M. Then M/N is also a generalized primal multiplication R-module.

Proof. Assume that L/N is a proper submodule of M/N and let K/N be a primal submodule of L/N. Then by Theorem 2.7, K is a primal submodule of L. Since M is a generalized multiplication R-module, we must have $K = (K :_R L)L$; hence $K/N = (K/N :_R L/N)(L/N)$, and the proof is complete. □

References

[1] Fuchs L., On primal ideals, Proc. Amer. Math. Soc., 1, 1950; 1-6.

[2] Fuchs L. and Mosteig E., Ideal theory in Pr\"ufer domains-An unconventional approach, J. Algebra, 252, 2002; 411-430.

[3] Dauns J., Primal modules, Comm. Algebra, 25(8), 1997; 2409-2435.

[4] Atani S.E., and Darani A.Y., Some remarks on primal submodules, To appear in Sarajevo Journal of Mathematics.

[5] Azizi A., Weak multiplication modules, Czechoslovak Math. J., 53(128),2003; 529-534.

[6] Macdonald I.G., Secondary representation of modules over commutative rings, Sympos. Math. XI,1973; 23-42.

[7] Sharp R.Y., Steps in Commutative Algebra, Cambridge University Press, Cambridge,1990.

[8] Lu C.P., Spectra of modules, Comm. Algebra, 23, 1995; 3741-3752.

[9] Atani S.E., On secondary modules over pullback rings, Comm. Algebra, 30(6), 2002; 2675-2685.

[10] McCasland R.Y. and Moore M. E., Prime Submodules, Comm. Algebra, 20(6), 1992; 1803-1817.

[11] Smith P.F., Some remarks on multiplication modules, Arch. Math., 50, 1998; 223-235.

[12] Atani S.E., Strongly irreducible submodules, Bull. Korean Math. Soc., 42, 2005; 121-131.