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Estimation of Functional Relationship Model for Circular Variables and Its Application in Measurement Problems

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ABSTRACT

This paper discusses on the functional relationship model for circular variables which may be used in the comparing two sets of circular data subjected to measurements or unobservable errors, for instance the measurements of wind direction by two different techniques or at two different levels. As opposed to the functional relationship model for linear or real line data, the proposed model is rather complicated. However, parameter estimation may be obtained numerically using iteration procedure. Further differences observed between the functional model for real line data sets are highlighted in the estimation process. Model verification and its behavior will be examined via simulation. The model is illustrated with an application to the analysis of wind direction data that have been collected from Bayan Lepas Airport, Malaysia measured at two different levels.

Keywords and phrases: Functional relationship model, circular variables, von Mises distribution, concentration parameters, circular residuals.

1. INTRODUCTION

The functional relationship model is part of the general class of error-in-variables model (EIVM), in which the underlying variables are deterministic (or fixed). If the variables are random, the model is known as a structural relationship model. The linear ultrastructural relationship model is the synthesis of the linear functional and structural relationship model. When the data is linear (i.e. takes values on the real line, as in the case of wind speed), an adequate statistical method for fitting a linear functional relationship model are described [1, 2]. We represent the observations and underlying variables generically by (x, y) and (X, Y) respectively. Some of the main differences between ordinary regression and EIVM can be summarized as follows: firstly, for any pair of observation (x, y) ordinary regression assumes the x value is mathematical observed without error, and only y is observed with error, whereas in EIVM both x and y are observed with error. Secondly, in EIVM there is no distinction between "explanatory" and "response" variables, unlike ordinary linear regression. Lastly, ordinary linear regression is more appropriate if the aim is to predict one variable from the other rather than to look at the underlying relationship between the two variables [3].

In some practical problems the variables are no longer linear, for instance to compare the measurements of wind direction using two different instruments. In this case the variables are known as circular variables, where the variables are taking values between (0, 2π] radians or (0°, 360°]. Due to the bounded closed space of the circular variables, different statistical techniques from those appropriate for linear variables must be used. Such data is encountered in many fields, as an example in geosience [4], neouroscience [5] and psychology [6].

The functional relationship model for circular variables was first introduced by extending the regression model for circular variables [7]. The aims of this paper are to highlight various issues with regards to the functional model for circular variables and its application in measurement problems. Based on simulation results, the significant differences from functional model for linear or real line data are highlighted.

2. THE MODEL

Suppose x_i and y_i are the observed values of the circular variables X and Y respectively, thus $0 \leq (x_i, y_i) < 2\pi$, for $i = 1,...,n$. The circular variables X and Y are true values corresponding to x_i and y_i respectively and there are linear relationship between these two variables. For any fixed X_i, we assume that the observations x_i and y_i have been measured with errors δ_i and ε_i respectively. Since the data are concentrated on $[0,2\pi)$ and linear combination of angles should also be $[0,2\pi)$, we need to use addition modulo 2π. Thus, the full model can be written as

$$x_i = X_i + \delta_i \text{ and } y_i = Y_i + \varepsilon_i \text{ and also}$$

$$Y_i = \alpha + \beta X_i \pmod{2\pi}, \text{ for } i = 1,2,...,n.$$

Parameter α is known as rotation parameter and it is a circular variables, whereas β is a real value close to unity assuming both variables are the measurements of the same properties but by different techniques. For practical purposes and to apply the model to such calibration problem, a restriction of β close to unity is appropriate, though the likelihood for a finite set of points may have a higher maximum at a large value of β well away from the neighbourhood of unity. This has been discussed in details by Hussin [7].

For above model, we also assume δ_i and ε_i are independently distributed of von Mises distribution with zero mean and concentration parameter κ and ν that is, $\delta_i \sim VM(0, \kappa)$ and $\varepsilon_i \sim VM(0, \nu)$ respectively, as opposed to the functional model for linear variables where the random errors are normally distributed. For a circular random variables θ, the von Mises probability distribution function (e.g. Rao *et al.* [8]) is given by,

$$g(\mu_0, \kappa; \theta) = \frac{1}{2\pi I_0(\kappa)} \exp\{\kappa \cos(\theta - \mu_0)\}$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero. The parameter μ_0 is the mean direction while the parameter κ is described as the concentration parameter.

As a comparison, the reciprocal of the concentration parameters, κ^{-1}, influences the von Mises distribution in essentially the same way σ^2 influence the normal distribution of $N(\mu, \sigma^2)$. Thus a concentrated von Mises distribution will have large concentration parameters and a dispersed von Mises distribution give a small concentration parameter. Another similarity to the normal distribution is that, for large κ, a random variable $\delta_i \sim VM(0,\kappa)$ is approximately distributed as $N(\mu, \kappa^{-1})$, [8]. This is important later in explaining the behavior of the maximum likelihood estimate of parameters. Further discussion on concentration parameter is also given by Khanabsakdi [9].

The log likelihood function for above model may be given by

log $L(\alpha,\beta,\kappa,\nu,X_1,...,X_n\,;\,\lambda,\,x_1,...,\,x_n,\,y_1,...,y_n)$ = $-2n\log(2\pi)-n\log I_0(\kappa)-n\log I_0(\nu)+\kappa\Sigma\cos(x_i-X_i)$ $+\nu\Sigma\cos(y_i-\alpha-\beta X_i)$

For estimation purposes, we assume the ratio of the error concentration parameters, which is $\lambda=\frac{\nu}{\kappa}$ is known or $\nu=\lambda\kappa$, a similar approach to the one used in the functional model for real line data, [1].

Hence, the log likelihood function is reduced to,

log $L(\alpha,\beta,\kappa,X_1,...,X_n\,;\,\lambda,\,x_1,...,\,x_n,\,y_1,...,y_n)$ =

$-2n\log(2\pi)-n\log I_0(\kappa)-n\log I_0(\lambda\kappa)+\kappa\Sigma\cos(x_i-X_i)+\lambda\kappa\Sigma\cos(y_i-\alpha-\beta X_i)$

There are $(n+3)$ parameters to be estimated, which are α,β,κ and X_i. Differentiating log L with respect to parameters α,β,κ and X_i, we can obtain $\hat{\alpha}$, $\hat{\beta}$ and $\hat{X}_i$. The first partial derivatives of the log likelihood function with respect to α is

$$\frac{\partial \log L}{\partial \alpha}=\sum \sin(y_i-\alpha-\beta X_i).$$

Setting this equal to zero and simplifying we get

$$\sum \sin(y_i-\hat{\beta}\hat{X}_i)\cos\hat{\alpha}-\sum \cos(y_i-\hat{\beta}\hat{X}_i)\sin\hat{\alpha}=0$$

This gives,

$$\tan\hat{\alpha}=\frac{\sum \sin(y_i-\hat{\beta}\hat{X}_i)}{\sum \cos(y_i-\hat{\beta}\hat{X}_i)}$$

$$\hat{\alpha}=\tan^{-1}\left\{\frac{\sum \sin(y_i-\hat{\beta}\hat{X}_i)}{\sum \cos(y_i-\hat{\beta}\hat{X}_i)}\right\}$$

$$=\tan^{-1}\left\{\frac{S}{C}\right\},\quad \text{say}.$$

That is,

$$\hat{\alpha}=\begin{cases}\tan^{-1}\left(\frac{S}{C}\right), & S>0, C>0\\ \tan^{-1}\left(\frac{S}{C}\right)+\pi, & C<0\\ \tan^{-1}\left(\frac{S}{C}\right)+2\pi, & S<0, C>0\end{cases}$$

The first partial derivative with respect to X_i is

$$\frac{\partial \log L}{\partial X_i} = \kappa \sin(x_i - X_i) + \lambda\kappa\beta \sin(y_i - \alpha - \beta X_i).$$

If we set this equal to zero, we may solve iteratively for X_i given some "initial guesses". Suppose $\hat{X}_{i0}$ is an initial estimate for $\hat{X}_i$. We write

$$x_i - \hat{X}_i = x_i - \hat{X}_{i0} + \hat{X}_{i0} - \hat{X}_i = (x_i - \hat{X}_{i0}) + \Delta_i,$$

where $\Delta_i = \hat{X}_{i0} - \hat{X}_i$, and also we have $y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i = (y_i - \hat{\alpha} - \hat{\beta}\hat{X}_{i0}) + \hat{\beta}\Delta_i$. Hence the above partial derivative equation becomes,

$$\sin(x_i - \hat{X}_{i0} + \Delta_i) + \lambda\hat{\beta}\sin(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_{i0} + \hat{\beta}\Delta_i) = 0.$$

For small Δ_i, we have cos $\Delta_i \approx 1$, cos $\hat{\beta}\Delta_i \approx 1$, sin $\Delta_i \approx \Delta_i$ and sin $\hat{\beta}\Delta_i \approx \hat{\beta}\Delta$. Hence the equation is simplified (approximately) to

$$\hat{X}_{i1} \approx \hat{X}_{i0} + \frac{\sin(x_i - \hat{X}_{i0}) + \lambda\hat{\beta}\sin(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_{i0})}{\cos(x_i - \hat{X}_{i0}) + \lambda\hat{\beta}^2\cos(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_{i0})}, \tag{1}$$

where $\hat{X}_{i1}$ is an improvement of $\hat{X}_{i0}$.

The first partial derivative with respect to β is

$$\frac{\partial \log L}{\partial \beta} = \sum X_i \sin(y_i - \alpha - \beta X_i).$$

$\hat{\beta}$ may also be obtained iteratively. Suppose $\hat{\beta}_0$ is an initial estimate of $\hat{\beta}$. Then $y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i = (y_i - \hat{\alpha} - \hat{\beta}_0\hat{X}_i) + \Delta\hat{X}_i$, where $\Delta = \hat{\beta}_0 - \hat{\beta}$. For small Δ, setting the partial derivative to zero gives

$$\hat{\beta}_1 \approx \hat{\beta}_0 + \frac{\sum \hat{X}_i \sin(y_i - \hat{\alpha} - \hat{\beta}_0\hat{X}_i)}{\sum \hat{X}^2{}_i \cos(y_i - \hat{\alpha} - \hat{\beta}_0\hat{X}_i)} \tag{2}$$

where $\hat{\beta}_1$ is an improvement of $\hat{\beta}_0$.

Possible initial estimates for iteration are $\hat{X}_{i0} = \hat{X}_i$ in Equation (1) for $i = 1,...,n$ and $\hat{\beta}_0 = 1.0$ in Equation (2). This is sensible since both the X and Y are estimates of the same quantity (i.e. direction), so 1.0 would be a logical initial guess of β, and x_i is a direct measurement of X_i. By using Equations (1) and (2) we can update α, β and X_i, and proceed iteratively. This iteration procedure will continue until the convergence criterion is satisfied.

Finaly, by setting $\frac{\partial \log L}{\partial \kappa} = 0$, we get the equation

$$A(\kappa) + \lambda A(\lambda\kappa) = \frac{1}{n}\left\{\sum \cos(x_i - \hat{X}_i) + \lambda \sum \cos(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i)\right\} \tag{3}$$

From the asymptotic power series for the Bessel functions $I_0(\kappa)$ and $I_1(\kappa)$, we have,

$$A(\kappa) = \frac{I_1(\kappa)}{I_0(\kappa)} = 1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3} + 0(\kappa^{-4}) \qquad (4)$$

Simplifying Equations (3) and (4) we have the expression approximately given by

$$8(1+\lambda-c)\kappa^3 - 8\kappa^2 - (1+\frac{1}{\lambda})\kappa - (1+\frac{1}{\lambda^2}) = 0, \qquad (5)$$

where

$$c = \frac{1}{n}\left\{\sum \cos(x_i - \hat{X}_i) + \lambda \sum \cos(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i)\right\}.$$

The above cubic equation in κ, i.e. Equation (5) has only one positive real root and two complex roots, giving $\hat{\kappa}$ as the positive real root.

Further, by assuming $\lambda = 1$, from Equation (3)

$$\hat{\kappa} = A^{-1}\left[\frac{1}{2n}\left\{\sum \cos\left(x_i - \hat{X}_i\right) + \sum \cos\left(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i\right)\right\}\right]$$

Estimation of κ can be obtained by using the approximation given by Fisher [10],

$$A^{-1}[w] = \begin{cases} 2w + w^3 + \frac{5}{6}w^5 & w < 0.53 \\ -0.4 + 1.39w + \frac{0.43}{(1-w)} & 0.53 \le w < 0.85 \\ \frac{1}{w^3 - 4w^2 + 3w} & w \ge 0.85 \end{cases}$$

Hence,

$$\hat{\kappa} = A^{-1}[w] \text{ where } w = \frac{1}{2n}\left\{\sum \cos\left(x_i - \hat{X}_i\right) + \sum \cos\left(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i\right)\right\}$$

Since in terms of the normal distribution model, $\hat{\sigma}^2 \approx \sigma^2/2$, or $1/\hat{\kappa} \approx 1/(2\kappa)$, we should have $\hat{\kappa} \approx 2\kappa$ in the von Mises distribution model, which suggest that a consistent estimator $\hat{\kappa}$ of κ in the functional relationship model for circular variables should be obtained by setting $\tilde{\kappa} = \hat{\kappa}/2$. Note that, in the next sections on simulation study and application of model, $\tilde{\kappa}$ gives indeed a good approximation to the value of κ.

By using various approximations and Fisher information matrix [11], we can find the estimated variance of parameters. In particular, by assuming equal error concentration parameter, it can be shown that

$$\hat{Var}(\tilde{\kappa}) = \frac{\tilde{\kappa}}{2n\left[\tilde{\kappa} - \tilde{\kappa}A^2(\tilde{\kappa}) - A(\tilde{\kappa})\right]}$$

$$\hat{Var}(\hat{\alpha}) = \frac{\left(1+\hat{\beta}^2\right)\sum \hat{X}_i^2}{\tilde{\kappa}A(\tilde{\kappa})\left[n\sum \hat{X}_i^2 - \left(\sum \hat{X}_i\right)^2\right]}$$

$$\hat{Var}(\hat{\beta}) = \frac{\left(1+\hat{\beta}^2\right)n}{\tilde{\kappa}A(\tilde{\kappa})\left[n\sum \hat{X}_i^2 - \left(\sum \hat{X}_i\right)^2\right]}$$

In this model, Equations (1) and (2) are nonlinear and it can be quite complicated to derive the expressions of its sample moments. In view of this, this paper does not study the biasness and consistency of the estimators of the model and its variances analytically.

3. Simulation Study

Simulation studies have been carried out in order to assess the accuracy and biasness of the proposed model. Sample sizes of n were generated and let s be the number of simulations. It is noted that, in this model, α is a circular variable while β and κ are continuous. The following computations were carried out in the simulation study.

(a) Biasness for α

i) The mean of circular parameter $\hat{\alpha}$, i.e $\bar{\hat{\alpha}}$

$$\bar{\hat{\alpha}} = \begin{cases} \tan^{-1}\left(\dfrac{S}{C}\right) & S>0, C>0 \\ \tan^{-1}\left(\dfrac{S}{C}\right)+\pi & C<0 \\ \tan^{-1}\left(\dfrac{S}{C}\right)+2\pi & S<0, C>0 \end{cases}$$

where $C = \sum \cos(\hat{\alpha}_j)$, and $S = \sum \sin(\hat{\alpha}_j)$.

ii) Circular Distance, $d = \pi - \left|\pi - \left|\bar{\hat{\alpha}} - \alpha\right|\right|$

iii) Mean resultant length, $R = \frac{1}{s}\sqrt{\left(\sum \cos(\hat{\alpha}_j)\right)^2 + \left(\sum \sin(\hat{\alpha}_j)\right)^2}$

(b) Biasness for β and κ

Let w be a generic term for **β and κ**. Then

i) Mean, $\bar{\hat{w}} = \frac{1}{s}\sum \hat{w}_j$

ii) Estimated Bias, $EB = \bar{\hat{w}} - w$

iii) Estimated Root Mean Square Errors, $ERMSE = \sqrt{\frac{1}{s}\sum\left(\hat{w}_j - w\right)^2}$

The simulation results with s = 5,000 for each set of parameter value are shown in Tables 1, 2 and 3. The values of X have been chosen from $VM\left(\frac{\pi}{4},3\right)$ and without loss of generality we choose $\alpha = 0$ and $\beta = 1$. Three different choices of concentration parameters κ = 15, 30 and 50 for random error assuming $\kappa = \nu$, and for each of these four choice of sample size n = 30, 70, 100, 500 has been considered. The values of κ cover a more realistic range as we expect the random error of circular variable is less dispersed.

It appears from Table 1 that $\hat{\alpha}$ is a good estimator of α. Its circular distance, d, which represents the biasness is generally decreases with the increases of sample size n and the concentration parameters of circular random error. Similar trend also can be observed for the mean resultant length, R, which suggests good accuracy as the value is close to one.

Similar conclusions may also be drawn from Table 2 as the mean of $\hat{\beta}$ close to the true value of β with the increase of sample size n and the concentration parameters, κ of circular random error. The similar trend is also observed to the estimate bias (EB) and root mean square error (ERMSE) which suggest that $\hat{\beta}$ is a good estimator of β.

Table 1. Simulation Results for $\hat{\alpha}$ (True value of α = 0.0 and β = 1.0).

Performance Indicator	Sample size, n	κ = 15	κ = 30	κ = 50
Circular Mean	30	6.2706	6.2763	6.2786
	70	6.2747	6.2802	6.2816
	100	6.2774	6.2804	6.2812
	500	6.2811	6.2821	6.2824
Circular Distance, d	30	0.0126	0.0069	0.0046
	70	0.0085	0.0030	0.0016
	100	0.0058	0.0028	0.0020
	500	0.0021	0.0011	0.0008
Mean Resultant Length, R	30	0.9955	0.9979	0.9987
	70	0.9982	0.9992	0.9995
	100	0.9987	0.9994	0.9997
	500	0.9997	0.9999	0.9999

Table 3 shows the simulation results for biasness of $\tilde{\kappa}$ for different sample size n = 30, 70, 100 and different error concentration parameter, κ. The results clearly indicates that division by 2 indeed is the right correction for the real maximum likelihood estimator. A more careful examination of the table reveals that the mean of $\tilde{\kappa}$ gets closer to the true value of κ with the increase of sample size n and the concentration parameters of circular random error. Thus, it seems that the larger κ (less dispersed), the better is the approximation of $\tilde{\kappa}$ to the real maximum likelihood estimator and the more appropriate is the correction $\tilde{\kappa} = \hat{\kappa}/2$.

Table 2. Simulation Results for $\hat{\beta}$ (True value of $\alpha = 0.0$ and $\beta = 1.0$).

Performance Indicator	Sample size, *n*	$\kappa = 15$	$\kappa = 30$	$\kappa = 50$
Mean	30	1.0085	1.0049	1.0027
	70	1.0044	1.0023	1.0012
	100	1.0038	1.0016	1.0012
	500	1.0012	1.0007	1.0006
Estimate Bias (EB)	30	0.0085	0.0049	0.0027
	70	0.0044	0.0023	0.0012
	100	0.0038	0.0016	0.0012
	500	0.0012	0.0007	0.0006
Estimate Root Mean square Error (ERMSE)	30	0.0482	0.0352	0.0272
	70	0.0273	0.0189	0.0149
	100	0.0231	0.0156	0.0123
	500	0.0103	0.0068	0.0054

Table 3. Simulation Results for $\tilde{\kappa}$ (True value of $\alpha = 0.0$ and $\beta = 1.0$).

Performance Indicator	Sample size, *n*	$\kappa = 15$	$\kappa = 30$	$\kappa = 50$
Mean	30	16.9581	34.3871	57.0958
	70	15.7002	31.6497	52.7753
	100	15.3441	31.0460	51.8156
Estimate Bias (EB)	30	1.9581	4.3871	7.0958
	70	0.7002	1.6497	2.7753
	100	0.3441	1.0460	1.8156
Estimate Root Mean square Error (ERMSE)	30	5.2567	10.8714	17.7250
	70	2.8582	5.8608	9.7104
	100	2.2397	4.5698	7.7546

4. Application to Measurement of Directional Data

As an illustration of the model, we considered the measurement of wind direction data that have been collected from Bayan Lepas Airport Malaysia, measured at two different levels. Data was recorded on July and August 2005 at 12:00 AM, located at

16.3 m above ground level, latitude 05°18′ N and longitude 100°16′ E. A total of 62 observations have been recorded at two different pressures which is at pressure 850 Hpa (with 5,000 m height as variable x) and at pressure 1,000 Hpa (with 300 m height as variable y).

The scatter plot for the raw data with line $y = x$ as shown in Figure 1 suggests the linearity assumption between the two variables of the data set. A few points seem to be outliers at the top left or bottom right of the plot. However, they are actually consistent with the rest of the observations as they are close to other observations at the top right or bottom left due to wrap-around measurement from 2π back to 0.

Table 4 shows the estimates of parameters and their standard error. Hence, the estimate relationship for Bayan Lepas data set is given by

$$Y = 0.464 + 0.907X \pmod{2\pi}$$

To check the goodness-of-fit for our

Table 4. Parameter estimation for Bayan Lepas Airport, Malaysia data set.

Parameter	Estimate	Standard Error
$\hat{\alpha}$	0.464	3.959×10^{-1}
$\hat{\beta}$	0.907	8.703×10^{-2}
$\hat{\kappa}$	1.263	2.769×10^{-2}

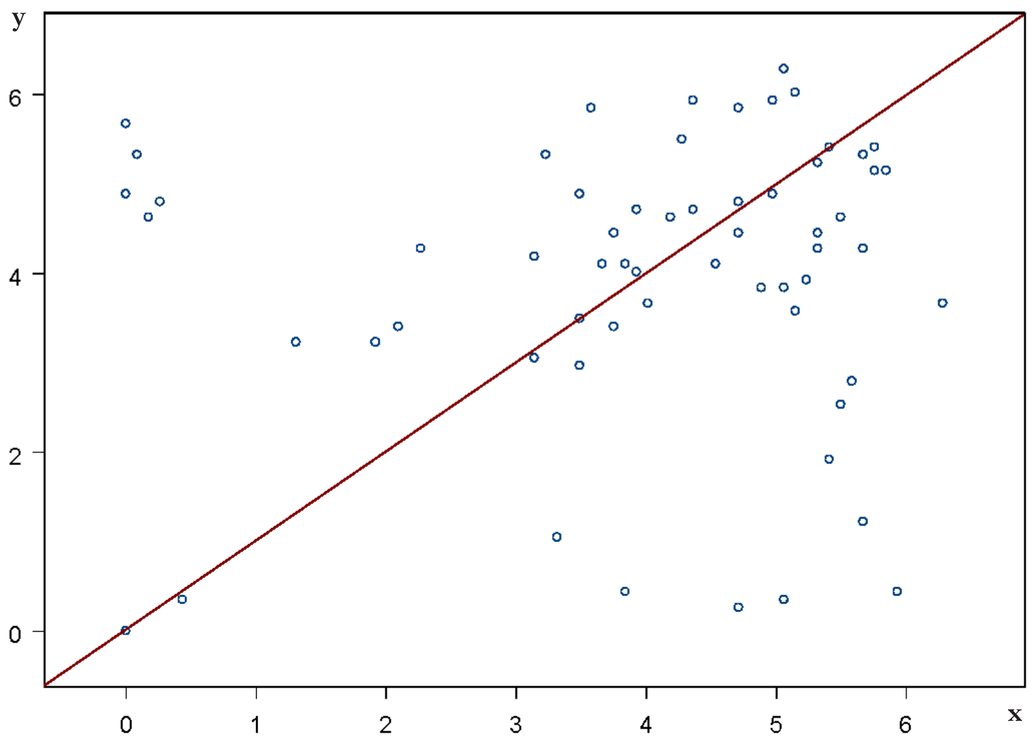

Figure 1. Scatter Plot for wind direction at Bayan Lepas Airport, Malaysia.

model, firstly, we use graphical method which is by plotting the von Mises quantile plot for the circular residuals of data set as shown in Figure 2.

From Figure 2, it is obvious that the data are scattered along the straight line which shows that the circular residuals satisfy our initial assumption that each circular residuals are independently distributed and followed the von Mises distribution.

Apart from graphical representation, we may also used Watson U^2 test and Kuiper's V test [9, 10], performed by ORIANA software [12] to calculate the value of U^2 and V respectively for the circular residuals, as given in Table 5.

From the values shown in Table 5, we can see that both the U^2 and V test for each δ and ε respectively are small and the null hypothesis that the samples are drawn from the von Mises distribution is not rejected. Hence, these imply that the samples are drawn from the von Mises distribution.

Table 5. Watson U^2 and Kuiper's V test for circular residuals.

Type of test and circular residuals		Test statistics	p-value
Watson U^2	δ	0.059	$0.25 < p < 0.5$
	ε	0.03	$p > 0.5$
Kuiper's V	δ	1.042	$p > 0.15$
	ε	0.886	$p > 0.15$

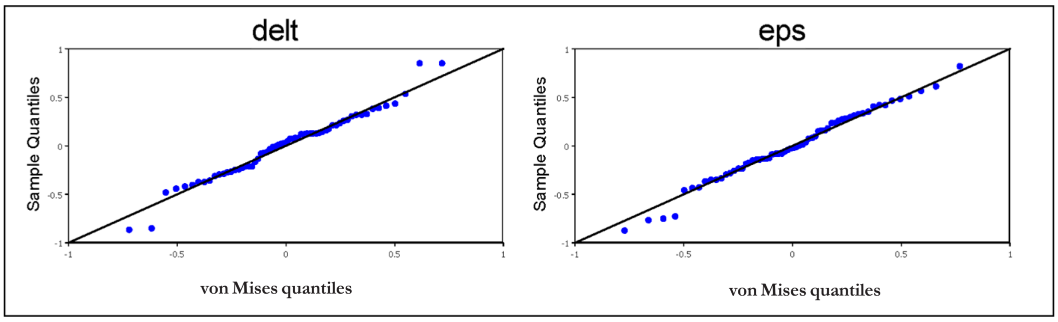

Figure 2. The von Mises quantiles plot of circular residuals.

5. Discussion

This paper suggests the functional relationship model for circular variables. The model is an analogue of the functional relationship model for continuous or real line data but assumes the random errors distributed as von Mises. Parameter estimation has been obtained by maximum likelihood assuming equal of errors concentration parameter, in which the same assumption is being imposed for the functional relationship model for real line data set. Estimation has been obtained iteratively since the closed-form expression for each estimate is not available, by choosing a suitable starting value. It is also suggested that from the simulation studies, the maximum likelihood of estimate gives consistent estimate for all the parameters. By using various approximations and Fisher information matrix, the variance and covariance matrix of the estimates are obtained. A significant finding from the study suggest that in the case of the functional relationship model for real line data, it is the estimator of a variance that need to be corrected by multiplying it by 2, but in the case of circular variance, it is the estimator of a concentration parameter (whose inverse is equivalent of the variance for real line data) that needs to be corrected by dividing it by 2. The model have been applied to the measurement problem by looking at the underlying relationship of wind direction at two different levels measured at Bayan Lepas Airport, Malaysia. It is found that, by assuming a known ratio of error concentration parameter, the proposed model reasonably represents the underlying relationship between the measurements of two circular variables.

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