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Contributed Paper

Small Convex Covers for Convex Unit Arcs

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ABSTRACT

More than 40 years ago, Leo Moser stated a geometry problem on the plane asking for the least-area set that contains a congruent copy of every unit arc. The smallest set known, by Norwood and Poole in 2004, has area 0.26043. One interesting related problem is to focus on a set that contains a congruent copy of every convex unit arc. Such a set is called a cover for convex unit arcs. The smallest cover known, by Wichiramala in 2005, has area 0.2464. In this work we establish a smaller cover for convex unit arcs of area less than 0.242.

Keywords: covering by convex sets, convex unit arc, worm problem.

1. INTRODUCTION

In 1966, Leo Moser stated a famous geometry problem on the plane [1]. The problem is to search for a set with least area that contains a congruent copy of every unit arc. It is called the worm problem of Leo Moser and is also called the Moser's worm problem.

Ability of a set to contain a congruent copy of a family of sets is a very interesting topic in classical geometry. One of the most famous problems was stated by Lebesgue, called "Lebesgue's universal covering problem", concerning sets on the plane. This problem asks for the smallest convex set, called a universal cover, that contain a copy of every set of unit diameter. Solving this problem completely seems to be out of reach. The main progress is about finding a universal cover smaller in area to the last. Another important progress is about finding a better lower bound for area of the smallest universal cover. Clearly Moser's worm problem has the same nature to the Lebesgue's problem. Moreover, the outcome is also the same. Before we mention progress of the worm problem, for convenient, we will define a way to explain the relation between sets on the plane as follows. We say that a set A can cover a set B if A contains a congruent copy of B, or in another word, there is an isometry f that $A \supseteq f(B)$. We say a set C is a cover for all sets in a family $\mathcal{F}$ of sets if C can cover every set in $\mathcal{F}$. A cover for a certain family of unit arcs may be called, in short, a worm cover. The very first discovery for worm covers were by Meir [2], by Wetzel [2] and by Gerriets [3] in 1970's. These covers have area 0.39270, 0.34501 and 0.3214 respectively. The smallest cover known with area 0.260437 is by Norwood and Poole [4]. One interesting aspect is that this cover is not convex. At the other end, the best lower bounds of the area of convex covers is 0.21946 [5, 6]. Along with the illusive quest for the answer, may variants are studied. One of the few most popular variants are looking for 1) smallest cover for convex unit arcs 2) smallest cover for closed unit arcs. All variants depend on 3 main parameters; 1) family of arcs to be covered 2) certain kinds of covers 3) specific way to cover. For example one may study simply connected sets that contain a translated copy of every three-segment unit arc. A simple lower bound of area of simply connected covers for closed unit arcs can be found as follows. Since a circle of perimeter 1 is a closed unit arc, its area $2\sqrt{\pi}$ is an obvious such lower bound. Key progress in finding small covers for convex unit arcs started in 1970's when Wetzel gave a stunning, short proof that the isosceles right triangle T_0 of hypothenuse 1 is a cover for convex arcs [7]. Its area is 0.25. Much later in 2002, a subset of T_0 with area 0.24655 obtained by clipping T_0 with 2 parabolas was presented by Johnson, Poole and Wetzel [7]. In 2003, Wichiramala introduced a smaller cover for convex arcs for the first time in the Geometric Potpourri Seminar [8]. One of the main ingredient is the use of numerical optimization in some cases of the proof. The numerical minimization is of the type called "convex programming" which is theoritically confirmed to reach the minimum with high precision.

In [7] Wetzel conjectured that a subset of T_0 of area 0.23982 is a cover for convex arcs. Later, Wichiramala proposed another conjecture that a similar subset of T_0 of the same area is also a cover [8]. In this work, we show that a quadrilateral similar to the one in [8] is a cover using a convex programming.

2. THE COVERS FOR CONVEX ARCS

Let T be the triangle with specification as illustrated by Figure 1 with $\alpha = 31.77^{\circ}$.

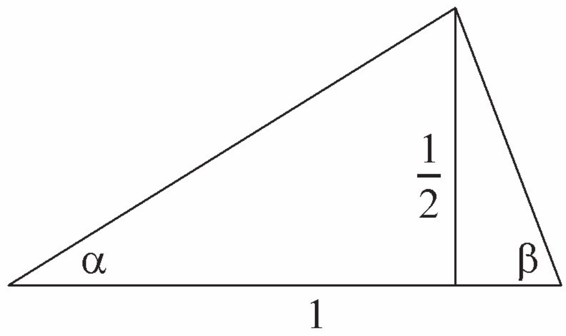

Figure 1. The triangle *T*.

Due to the condition $\frac{1}{2}\cot\alpha + \frac{1}{2}\cot\beta = 1$, we have $\beta = \operatorname{arccot}(2-\cot\alpha) = 68.92924^{\circ}$. The area of T is 0.5. We will show that T is a cover for convex unit arcs. Let $\overline{T}$ be the quadrilateral obtained by clipping T at its tip as illustrated by Figure 2 with $h_0 = 0.44378$ and $h_1 = 0.352324$.

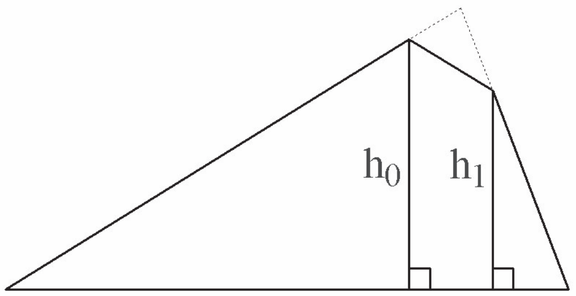

Figure 2. The quadrilateral $\overline{T}$.

The area of $\overline{T}$ is $\dfrac{h_0^2}{2\tan\alpha}+\dfrac{h_0+h_1}{2}\left(1-\dfrac{h_0}{\tan\alpha}-\dfrac{h_1}{\tan\beta}\right)+\dfrac{h_1^2}{2\tan\beta}=0.241698$.

On the plane, when we describe a congruent copy S' of a set S, we consider 3 parameters to refer to S' related to S. These parameters are location, orientation and rotation. In this fashion, a translated copy S'' of S has the same orientation and rotation as S. We say S'' and S are in the same ***post*** or ***pose*** the same way. When a set A has a translated copy contained in a set B, we say A can be ***located*** in B.

For convenience, we name copies of T and of $\overline{T}$ according to their orientations and rotations as follows. Let T_θ be a copy of T with the same orientation as T that can be obtained by rotating T for angle θ. Let $T^|$ be a copy of T obtained by reflecting T across a vertical line. Let $T^|_\theta$ be a copy of T obtained by rotating $T^|$ for angle θ. Note that $T = T_0$ and $T^| = T^|_0$. We name $\overline{T}_\theta$ and $\overline{T}^|_\theta$ in similar way.

The next lemma will tell a sufficient condition for a constrained two-segment arc to be shortest in total length of the 2 segments.

Lemma 1 *[8] On the plane, let l be a line and P and Q be 2 distinct points.*

1) If P and Q are on the same side of l, then there is only one point R on l that minimizes the sum of the distances $PR + RQ$. Furthermore, R is the only point for which the segments $\overline{PR}$ and $\overline{RQ}$ make the same angle with l.

2) If P and Q are on l or are not on the same side of l, then the set of all minimum points are the intersection of the segment PQ and l.

We will refer to the condition in the previous lemma as ***"the angle condition"***.

3. MAIN RESULT

In this section, we will show that T and $\overline{T}$ are covers for convex unit arcs. Throughout the proofs of the following 2 theorems, we fix an arc γ in its "standing post" where the line through its endpoints is horizontal and is under γ (see Figure 3).

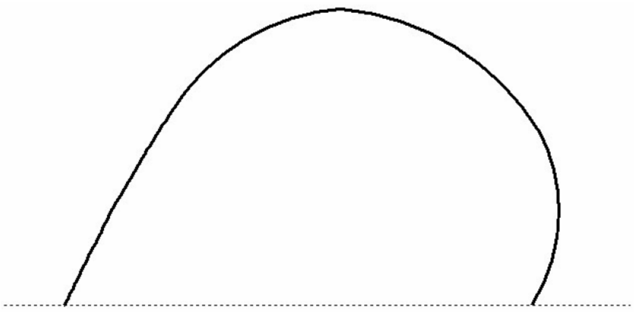

Figure 3. An arc in its "standing post".

Theorem 2. *T is a cover for convex unit arcs. In particular, every convex unit arc can be located in T or $T^{|}$ in its standing post.*

Proof. Let γ be a convex unit arc. Suppose to get a contradiction that γ cannot be located in T nor $T^{|}$ in its standing post. Let L_F and R_F be the left and right endpoints of γ. We can translate γ related to T so that L_F and R_F are on the ray A_1C_1 and γ touches the ray A_1B_1 (see Figure 4).

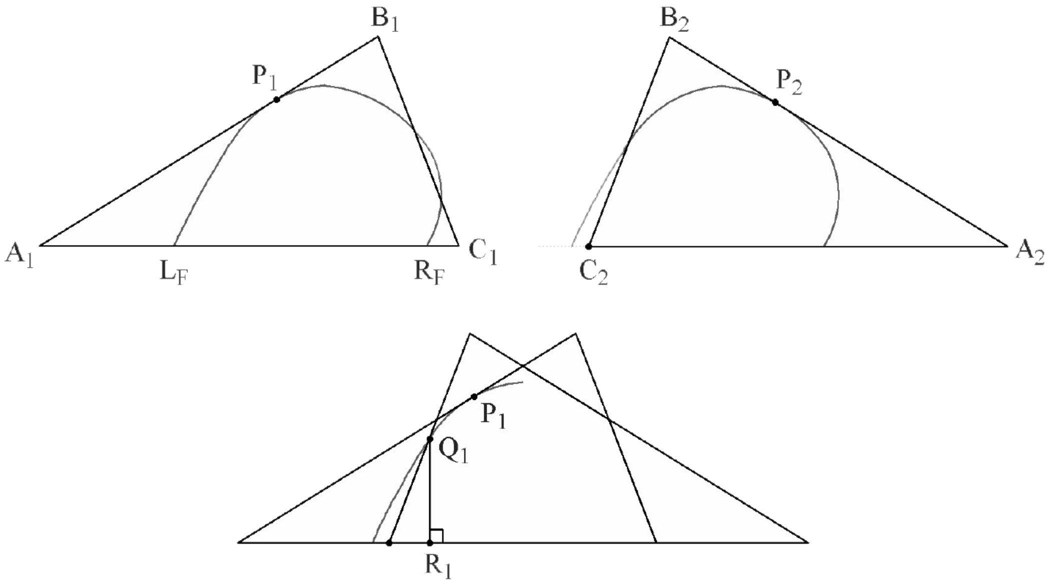

Figure 4. Translation of γ into T and $T^{|}$.

Let P_1 be the highest point γ touches the ray A_1B_1. Define P_2 in similar way as P_1. If P_1 is above the intersection of A_1B_1 and B_2C_2, let Q_1 be the highest point γ intersects B_2C_2. Otherwise, let Q_1 be P_1. Let R_1 be the point on A_1C_1 under Q_1. Define P_2, Q_2 and R_2 in similar way. We may assume that P_1 is not lower than P_2. Let π be the polysegment $R_1Q_1P_1P_2Q_2R_2$. We call π an admissible arc. Due to compactness of the set of all short admissible arcs, there exists a shortest such arc.

Case $P_1 \neq Q_1$ (see Figure 5). Since P_2 is not higher than P_1, the angle $B_1P_1P_2$ is at least α.

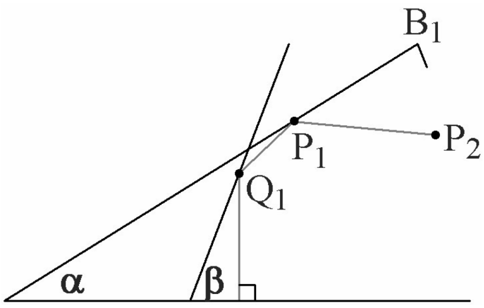

Figure 5. The arc π when $P_1 \neq Q_1$.

According to the angle condition, the arc π is not shortest among all admissible arcs.

Case $P_1 = Q_1$. Since P_2 is not higher than P_1, we have $P_2 = Q_2$. Hence π is the three-segment $R_1P_1P_2R_2$. A shortest admissible arc must be in this three-segment pattern. We will show that a shortest admissible arc must have a "staple" shape.

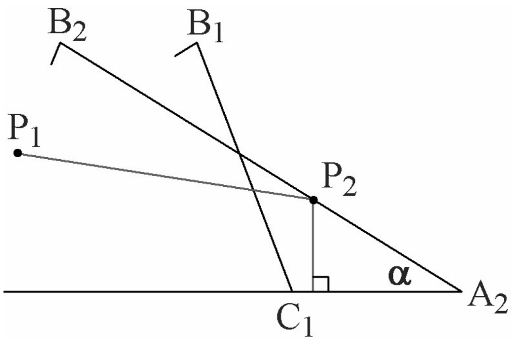

Figure 6. The arc π when $P_1 = Q_1$ and $P_2 \neq A_2B_2 \cap B_1C_1$.

Subcase P_2 is lower than the intersection of A_2B_2 and B_1C_1 (see Figure 6). Since the angle $B_2P_2P_1$ is at most α, the arc π is not shortest.

Subcase P_2 is the intersection of A_2B_2 and B_1C_1. Hence P_1 is the intersection of A_1B_1 and B_2C_2. Note that every shortest admissible arc is in this subcase. The arc π has three segment and fit in T with P_1P_2 horizontal and thus $l(\pi) = 1$. Since γ has a point on the left of B_2C_2, $l(\gamma) > l(\pi) = 1$.

In every case, we conclude that $l(\gamma) > 1$, a contradiction. Therefore the proof is complete.

Next we will prove the main result.

Theorem 3. $\overline{T}$ *is a cover for convex unit arcs.*

Proof. Suppose to get a contradiction that $\overline{T}$ cannot cover a convex unit arc γ. From the previous theorem, we may assume that γ can be located in T_0 in its standing post. Define L_E, R_E, L_S, R_S, L_K, R_K, L_A and R_A to be highest touching points to support lines parallel to sides of $\overline{T}$, $\overline{T}^{|}$ and $\overline{T}_{-90°-\frac{\alpha}{2}}$ illustrated in Figure 7. Note that E, S, K, A and F stand for ear, shoulder

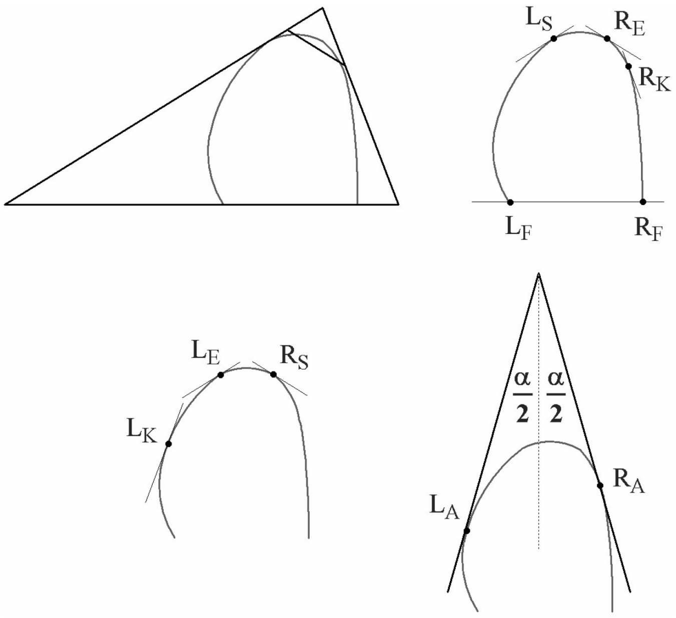

Figure 7. Special points according to support lines.

knee, ankle and foot. In this proof h_1 is chosen so that the angle between the lines BC and AD is α (see Figure 8). Thus $R_E = R_S$ and $L_E = L_S$.

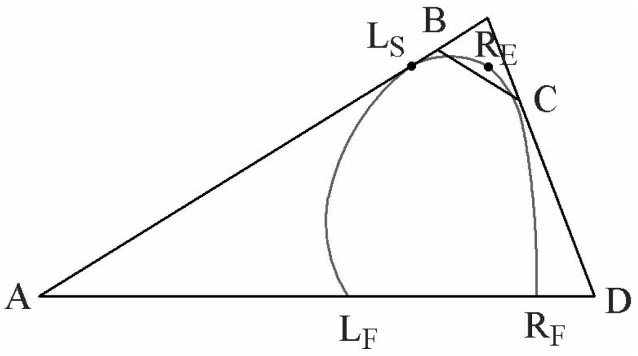

Figure 8. The arc γ and $\overline{T}_0$.

Due to the angles of support lines, these points appear on γ in this order: L_F - L_A - L_K - L_S = L_E - R_E = R_S - R_K - R_A - R_F . Let π be the polysegment of this order. Note that $l(\pi) \leq l(\gamma)$. We will show next that $l(\pi) > 1$.

Since γ cannot be located in $\overline{T}_0$, we can locate γ in T_0 so that (see Figure 8) L_F and R_F are on the ray AD, L_S is on the ray AB, and R_E is above the line BC (called condition c1.) Since γ cannot be located in $\overline{T}^{|}_0$, we can locate γ so that (see Figure 9) L_F and R_F are on the ray AD,

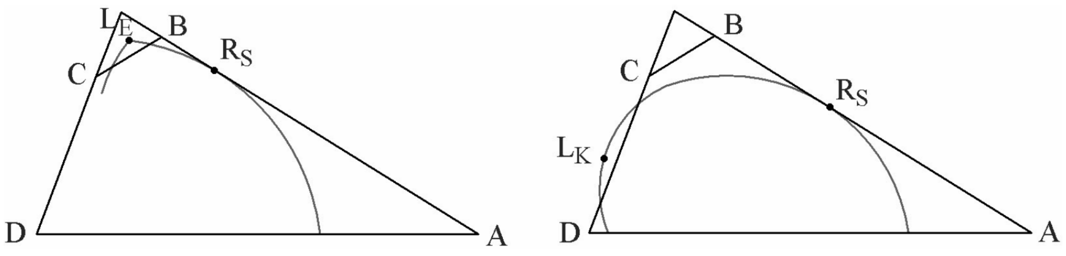

Figure 9. The arc γ and $\overline{T}^{|}$.

R_S is on the ray AB, and that L_E is above the line BC (called condition c2a) or L_K is above the line CD (called condition c2b). Since γ cannot be located in $\overline{T}_{-90^\circ-\frac{\alpha}{2}}$, we can locate γ so that (see Figure 10) L_A is on the ray AD, R_A is on the ray AB, and R_F is under the line BC (called

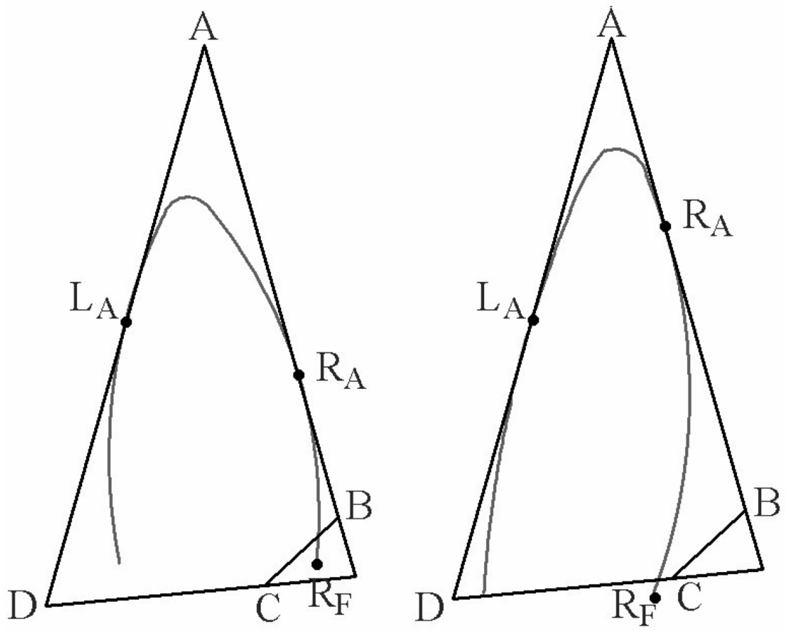

Figure 10. The arc γ and $\overline{T}_{-90^\circ-\frac{\alpha}{2}}$.

condition c3a) or the line CD (called condition c3b). Since γ cannot be located in $\overline{T}_{-90^\circ-\frac{\alpha}{2}}$, we can locate γ so that (see Figure 11) L_A is on the ray AB, R_A is on the ray AD,

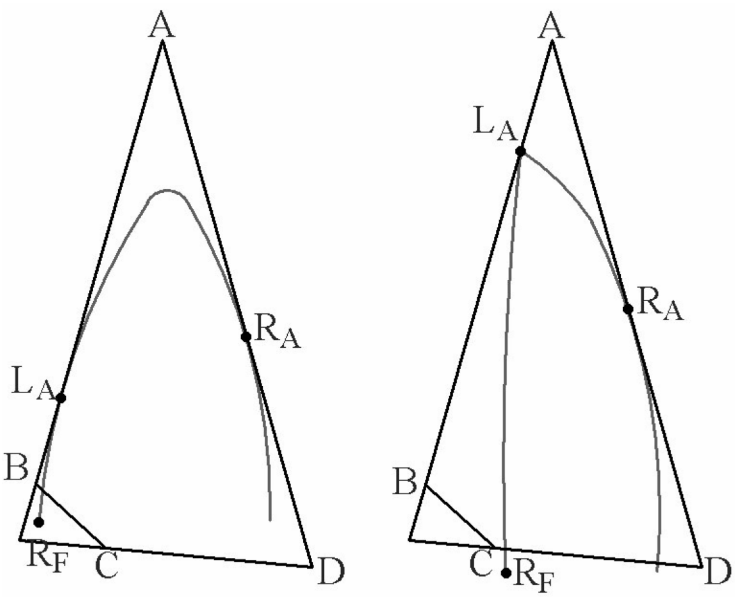

Figure 11. The arc γ and $\overline{T}_{90°+\frac{\alpha}{2}}$.

and L_F is under the line BC (called condition c4a) or the line CD (called condition c4b). Hence we can divide into cases depending on how γ satisfy conditions as follows.

Case 1. c1, c2a, c3a and c4a
Case 2. c1, c2a, c3a and c4b
Case 3. c1, c2a, c3b and c4a
Case 4. c1, c2a, c3b and c4b
Case 5. c1, c2b, c3a and c4a
Case 6. c1, c2b, c3a and c4b
Case 7. c1, c2b, c3b and c4a
Case 8. c1, c2b, c3b and c4b

The length of π is a convex function over positions of its vertices. Conditions on π narrows down possibilities of vertices to a convex domain. By a fundamental theorem in convex programming, the numerical minimization of the length is guaranteed to achieve in acceptable accuracy. From numerical minimization in appendix, we found in each case that

Case 1. $l(\pi) \geq 1.00006$.
Case 2. $l(\pi) \geq 1.01325$.
Case 3. $l(\pi) \geq 1.01324$.
Case 4. $l(\pi) \geq 1.0133$.
Case 5. $l(\pi) \geq 1.0001$.
Case 6. $l(\pi) \geq 1.01325$.
Case 7. $l(\pi) \geq 1.01325$.
Case 8. $l(\pi) \geq 1.01329$.

In every case, $l(\gamma) \geq l(\pi) \geq 1.00006$, a contradiction. Therefore the proof is complete.

From the previous theorem, we found that $\overline{T}$ is a cover for convex unit arcs that is smallest compared to all known covers of this kind. The cover $\overline{T}$ is convex and can be improved to be a little smaller by finding appropriate values of α, h_0 and h_1. It is also interesting to increase altitude and diameter of the triangle T before being clipped. However the 2 conjectures on covers for convex unit arcs by Wetzel in [7] and later by Wichiramala [8] are still open.

4. ACKNOWLEDGEMENTS

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5. APPENDIX: NUMERICAL MINIMIZATION FOR THE MAIN RESULT

First we mention the model for the coming program. We start with copies of $\overline{T}$ that are used in the proof, such as $\overline{T}_0$, $\overline{T}^{|}_0$, $\overline{T}_{-90°-\frac{\alpha}{2}}$ and $\overline{T}^{|}_{90°+\frac{\alpha}{2}}$. Important points are located as in Figure 12. Important points on γ are $L_F, L_S, L_K, L_E, R_E, R_S, R_K, R_A$ and R_F.

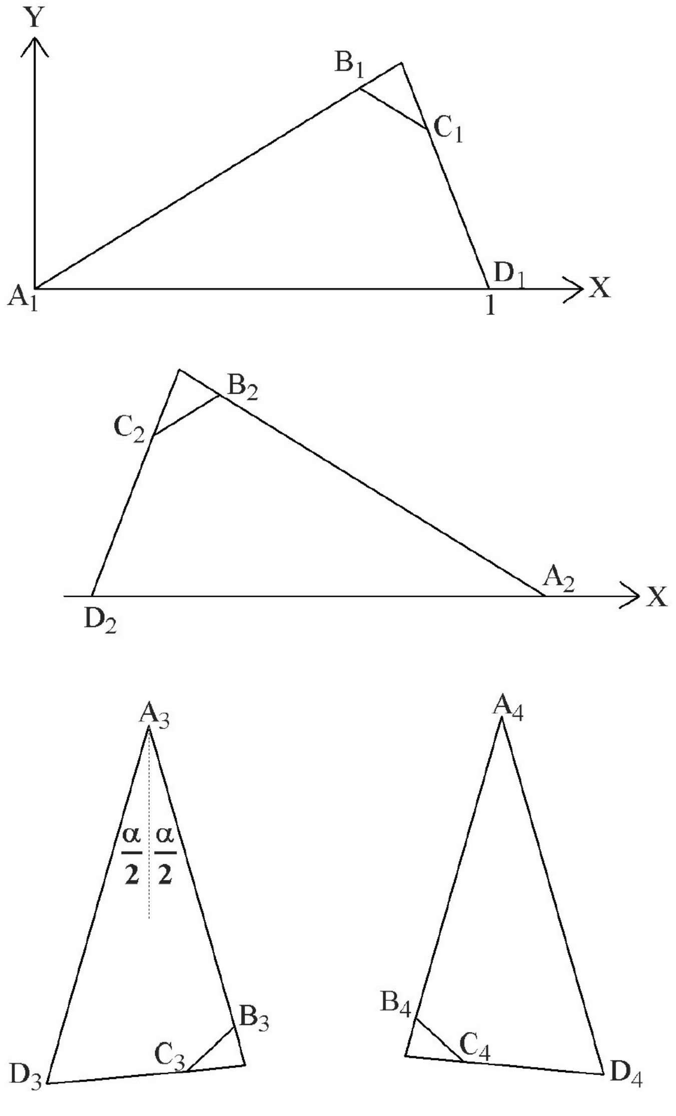

Figure 12. The 4 copies of $\overline{T}$.

The basic conditions on these points are

1. L_F and R_F are on X-axis.
2. L_S is on the ray A_1B_1.
3. R_S is on the ray A_2B_2.
4. L_A is on the ray A_3D_3.
5. R_A is on the ray A_4D_4 where $A_4 = A_3$.

The additional conditions are
Condition 1:

R_E is on the right of the ray $\overrightarrow{C_1B_1}$

Condition 2a:

L_E is on the right of the ray $\overrightarrow{B_2C_2}$

Condition 2b:

L_K is on the right of the ray $\overrightarrow{C_2D_2}$

Condition 3a:

R_F is on the right of the ray $\overrightarrow{C_3B_3}$

Condition 3b:

R_F is on the right of the ray $\overrightarrow{D_3C_3}$

Condition 4a:

L_F is on the right of the ray $\overrightarrow{B_4C_4}$

Condition 4b:

L_F is on the right of the ray $\overrightarrow{C_4D_4}$

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