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Contributed Paper

Two Rigid Spheres in Low-Reynolds Number-Gradient Flow

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ABSTRACT

When the effective viscosity of suspensions is modeled, the main gradient flow

$$\nu|_{\infty} \simeq V + Gx$$

is perturbed by the presence of spherical inclusions. Here V is the uniform stream, and G is the constant velocity gradient at infinity. The flow around a single sphere allows one to find the average contribution to the effective viscosity within the first order with respect to the volume fractions of the particulate phase. In order to obtain the second asymptotic order, one needs to solve the problem of the flow around two non-equal spheres under constant gradient at infinity, which is essentially a 3D problem.

In this study, the underlying symmetries of the flow are used, and the full 3D problem is reduced to five conjugated 2D problems. Each of these 2D problems is formulated in terms of stream functions which requite solving equations with bi-Stokesian operators. Bi-spherical coordinates are used for which the boundaries of the spheres are also coordinate surfaces. To solve the bi-Sotkesian equations, a fast spectral method based on Legendre polynomials is proposed with exponential convergence. The method of generating function is used for both Legendre and associated Legendre polynomials and closed algebraic systems are obtained for the systems under considerations.

Keywords: bi-Stokesian equation, spectral method, Associated Legendre polynomials.

1. INTRODUCTION

The hydrodynamic behavior of solid particles or fluid drops moving in a continuous medium at very small Reynolds numbers has a great importance for investigations in the fields of chemical, biochemical, and environmental engineering and science. The theoretical study of Stokes of the flow created by a translating rigid sphere in a viscous fluid has been extended by Hadamard in 1911 and Rybczynski in 1911 to the translation of a fluid sphere (droplet). In most practical applications, particles or drops are not isolated. Rather they interact through the disturbances that they introduce in the surrounding liquid. Hence, it is important to determine how the presence of neighboring particles affects the motion of the fluid inside and outside of droplets/particles, and determine their interaction. Estimating the effective transport coefficients of heterogeneous media is of great importance for many technological processes. The most typical examples of such media are suspensions, in which the second (particulate) phase is comprised by spherical particles (the filler) that are randomly dispersed throughout the continuous phase (the matrix). The different transport problems that can be considered for a suspension are the effective electric or heat conductivity, effective viscosity, and effective elasticity.

The first successful attempt to estimate the effective electric conductivity is due to Maxwell [1] who compared the potential created by n spheres of radius r each to the potential of a sphere that encompasses the swarm of small spheres and has an equivalent electric conductivity in the sense of yielding the same potential at a large distance. Using this approach, Maxwell obtained the contribution to the effective conductivity of first order with respect to the volume fraction of the particulate phase. The same idea was applied by Einstein [2] for computing the first-order in volume fraction contribution to the effective viscosity of a suspension. For the elastic moduli of a suspension, the same approach was applied in Walpole's paper [3].

Jeffrey [4] argued that the method proposed by Maxwell can give correctly only the first order in the volume fraction and went on to discuss the statistical properties of the centers of spheres. He extended the arguments of Batchelor and Green [5] and justified the conclusion that the second order approximation in the volume fraction can be obtained only if the solution around two spheres is obtained. A comprehensive review of the works on viscosity of suspensions can be found by Herczynski and Pienkovska [6].

The method of functional expansions (Volterra-Wiener series) with random point basis function (RPF) for rigorous treatment of the statistical properties of materials with random structure has originated in Christov's paper [7]. The application to estimating the effective heat conduction modulus of monodisperse suspension was presented in Christov and Markov's paper [8], while the elastic moduli were treated in Christov and Markov's paper [9]. After the generalization of the RPF expansion to marked random point functions was outlined [10], the most general case of a polydisperse suspension of perfect disorder type became amenable to the Volterra-Wiener method [11]. Nowadays, it can be considered as proven that the two-sphere solution does rigorously lead to the second-order approximation with respect to the volume fraction as it gives precisely the second-order kernel in the formal Volterra-Wiener expansion. Since the effective transport properties are aimed at, one needs to solve the two-sphere problem under constant gradient of the main field at infinity. This defines the main goal of the present paper: to develop an efficient numerical tool for solving the two-sphere problem.

A method to solve the Laplace equation, called currently "twin-pole expansion" [4] was proposed by Hicks [12]. The method consists in expanding the solution in spherical harmonics around two poles. This method was used on numerous occasions. Its main advantage lies in the fact that the integrals needed to compute the overall transport coefficients are easy to evaluate. For this reason, Jeffrey [4] went on to suggest that, in the context of the statistical theory of suspension, the twin-pole expansion is superior to the method involving bi-spherical coordinates. This claim is not immediately verifyable because the twin-pole expansion actually involves two levels of approximation: the first level is the truncation of the Legendre series. The second level of approximation stems from the fact that the functional coefficients of the series which depend on the radial coordinate, cannot be found in closed form. Rather, the solution is sought in asymptotic series with respect to the small parameter r/D, where r is the radius of the bigger of the spheres, and D is the distance between their centers.

The procedure of asymptotic solution can be interpreted physically as adding to the solution created by the boundary condition on one of the boundaries, a solution that is reflected from the other boundary. The procedure is also known as the "method of reflections" [13]. It has been successfully applied in various problems with two boundaries (e.g., for Stokes flow around a sphere in a cylindrical pipe [14]).

For the case of closely situated spheres when one of the radii is much greater than the other, the said parameter can actually tend to unity, which can make the respective series very slowly convergent. The bi-spherical coordinates offer an approach that is free of this limitation.

Without belittling the importance of the twin-pole expansion, a numerical solution with controlled convergence is still in demand, if for no other reason, but at least for estimating the region of convergence of the twin-pole expansion. The approach based on bi-spherical coordinates gives the solution in closed form, albeit in an infinite series with respect to the Legendre polynomials.

A successful numerical (e.g., spectral) solution is contingent on finding the appropriate curvilinear coordinates in which the boundaries of the domain of the solution are coordinate lines. The fact that the bi-spherical coordinates are the best suited tool for solving a transport problem in a medium containing two spherical inclusions was first emphasized by Lord Kelvin. He was apparently the first to introduce the bi-spherical coordinates in 1846 in a letter to Liouville [15]. The first detailed application of the bi-spherical coordinates for solving the Laplace equation was given by Jeffery [16] for the potential flow around two spheres.

The important difference of our research is that the flow stream has a constant gradient at infinity. The application of the bi-spherical coordinates to the heat conduction problem around two spheres with constant gradient at infinity was studied by Christov [17], where the Legendre-series method is worked out analytically but no numerical results were presented. An important advance in that paper was the proposed effective way to reduce an essentially 3D problem to a set of three 2D problems. The numerical results for this case have recently been obtained by Chowdhury and Christov [18]. They present the solutions for the temperature distribution with both longitudinal and transverse gradients at infinity, and demonstrated the very fast convergence of the Legendre-series method for problems of the type they considered.

Due to the specific dimension of the particles and the intraparticle distances (characteristic lengths) in suspensions, the particles can be considered as being very small and that their movements are very slow. As a result of these assumptions, the problems under consideration are, in fact, quasisteady, and the explicit dependence on time can be neglected in the equations. In this study we shall be interested in the problem of the gradient creeping (Stokes) flow around two spheres. Following the gist of the two works of Christov [17], and Chowdhury and Christov [18], we generalized the idea there, and succeeded in reducing the original 3D problem to five 2D problems. This radically reduces the complexity of the problem when treated numerically. Note that the flow of viscous liquid is mathematically speaking significantly more complicated than the problem of heat conduction, where the 3D problem was equivalent to just three 2D problems. The five systems are one of the important contributions of the present work. The decisions are made to focus only on the first of the system derived there, and it is recast in bi-spherical coordinates in terms of stream function.

In the following section the objective is to find a semi-analytical solution in Legendre series for the stream function from the first of these systems. The important contribution is that for the first time in the literature, a Legendre series technique is applied for solving equation with bi-Stokesian operator. We show how to obtain closed system for the unknown coefficients of the Legendre series for the stream function by satisfying the boundary conditions which are expanded into series with respect to associated Legendre polynomials.

2. STATEMENT OF THE PROBLEM

Consider two rigid spheres of generally unequal radii r_i, $i = 1,2$ suspended in an incompressible fluid of kinematic viscosity ν. Assuming creeping flow (very low Reynolds number), one can use the linear Stokes equations instead of the full Navier-Stokes equations. The boundary conditions include the non-slip condition on the sphere's boundaries, and the requirement for a constant velocity gradient at infinity. A sketch of the problem is shown in Figure 1.

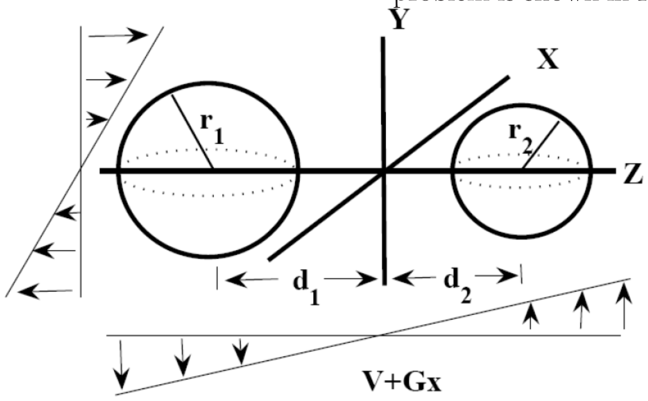

Figure 1. Gradient flow past two spheres.

For convenience, we recast the problem in terms of velocity perturbation $u = v - \tilde{V}$, where $\tilde{V} = V + Gx$. The governing equations for u adopt the forms

$$\frac{1}{\rho}\nabla p = \nu \nabla^2 u, \; x \in \Omega, \qquad \nabla \cdot u = 0, \; x \in \Omega, \tag{1}$$

with boundary conditions

$$\boldsymbol{u}|_{\Gamma} = -\tilde{\boldsymbol{V}}|_{\Gamma}, \qquad \boldsymbol{u}|_{x\to\infty} = 0, \tag{2}$$

where Γ is the composite boundary of the two inclusions (spheres), Ω is the region exterior to the spheres, V is the constant velocity vector at infinity, and G is a constant tensor of velocity gradient. In the above formula we use the notations x for the position vector, and u for the velocity vector, namely

$$\boldsymbol{x}=\begin{pmatrix}x\\ y\\ z\end{pmatrix},\ \boldsymbol{u}=\begin{pmatrix}u_x\\ u_y\\ u_z\end{pmatrix},\ \boldsymbol{V}=\begin{pmatrix}V^{(x)}\\ V^{(y)}\\ V^{(z)}\end{pmatrix},\ \boldsymbol{G}=\begin{pmatrix}G_{11} & G_{12} & G_{13}\\ G_{21} & G_{22} & G_{23}\\ G_{31} & G_{32} & G_{33}\end{pmatrix} \tag{3}$$

Note that the incompressibility condition imposes the following restriction on the components of the velocity gradient at infinity: $G_{11} + G_{22} + G_{33} = 0.$

3. Reduction of the 3-Dimensional B.V.P. to Sequence of Five 2-Dimensional B.V.P.'s

The important difference between the considered problem and other similar study is that the flow at infinity has a constant velocity gradient. This makes problem essentially 3D. An effective way to take advantage of the linearity of the problem, when dealing with the boundary conditions, was proposed in [17] for the case of the temperature distribution around two spheres with a constant gradient at infinity. The gist of that method is to take advantage of the fact that there is no explicit dependence on the polar angle in the Laplace equation (what is called a 'cyclic variable'), and the authors presented the sought solution as a linear combination of Fourier functions of the polar angle as dictated only by the boundary conditions. We generalize this idea to our case. The essential difference in our case is that there are five functions of the polar coordinates that enter the boundary conditions.

Let us first render problem (1)-(2) into cylindrical coordinates (r,ϕ,z)

$$x = r\cos\phi,\ \ y = r\sin\phi,\ \ z = z, \tag{4}$$

where $r \ge 0,\ \phi \in [0,2\pi],\ z \in (-\infty,\infty)$. Then equations (1) adopt the form

$$\nu\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}ru_r + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial\phi^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial^2 u_r}{\partial z^2}\right] = \frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{5}$$

$$\nu\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}ru_\phi + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\phi} + \frac{\partial^2 u_\phi}{\partial z^2}\right] = \frac{1}{\rho r}\frac{\partial p}{\partial\phi}, \tag{6}$$

$$\nu\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}u_z + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial\phi^2} + \frac{\partial^2 u_r}{\partial z^2}\right] = \frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{7}$$

$$\frac{1}{r}\frac{\partial}{\partial r}ru_r + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_r}{\partial z} = 0, \tag{8}$$

where u_r, u_ϕ, u_Z are the velocity components in terms of the cylindrical coordinates $u = (u_r, u_\phi, u_Z)$. In terms of the cylindrical coordinates, the boundary conditions on the sphere surfaces can be recast as follows

$$u_r\big|_{\Gamma} = -(G_{11} + G_{22})\frac{r}{2} - (V^{(x)} + G_{13}z)\cos\phi - (V^{(y)} + G_{23}z)\sin\phi$$
$$-(G_{11} - G_{22})\frac{r}{2}\cos 2\phi - (G_{12} + G_{21})\frac{r}{2}\sin 2\phi, \qquad (9)$$

$$u_\phi\big|_{\Gamma} = -(G_{21} - G_{12})\frac{r}{2} - (V^{(y)} + G_{23}z)\cos\phi + (V^{(x)} + G_{13}z)\sin\phi$$
$$-(G_{21} + G_{12})\frac{r}{2}\cos 2\phi - (G_{22} - G_{11})\frac{r}{2}\sin 2\phi, \qquad (10)$$

$$u_z\big|_{\Gamma} = -(V^{(z)} + G_{33}z) - G_{31}r\cos\phi - G_{32}r\sin\phi. \qquad (11)$$

Recall that at infinity we have $u_r = u_\phi = u_Z = 0$ and that the variable ϕ is a cyclic variable, i.e., it does not enter the coefficients of Eqs. (5)-(8). This means that the symmetry of the boundary conditions is entirely defined by Eqs. (9)-(11), which hints at the idea that one can seek the solutions of the 3D problems in the form of the following linear combinations

$$u_r(r,\phi,z) = u_r^{(0)}(r,z) + u_r^{(1)}(r,z)\cos\phi + u_r^{(2)}(r,z)\sin\phi + u_r^{(3)}(r,z)\cos 2\phi + u_r^{(4)}(r,z)\sin 2\phi, \qquad (12)$$

$$u_z(r,\phi,z) = u_z^{(0)}(r,z) + u_z^{(1)}(r,z)\cos\phi + u_z^{(2)}(r,z)\sin\phi + u_z^{(3)}(r,z)\cos 2\phi + u_z^{(4)}(r,z)\sin 2\phi, \qquad (13)$$

$$u_\phi(r,\phi,z) = u_\phi^{(0)}(r,z) + u_\phi^{(1)}(r,z)\cos\phi + u_\phi^{(2)}(r,z)\sin\phi + u_\phi^{(3)}(r,z)\cos 2\phi + u_\phi^{(4)}(r,z)\sin 2\phi, \qquad (14)$$

$$p(r,\phi,z) = p^{(0)}(r,z) + p^{(1)}(r,z)\cos\phi + p^{(2)}(r,z)\sin\phi + p^{(3)}(r,z)\cos 2\phi + p^{(4)}(r,z)\sin 2\phi. \qquad (15)$$

Since the Stokes equations are linear and the functions 1, $\cos\phi$, $\sin\phi$, $\cos 2\phi$ and $\sin 2\phi$ are linearly independent, the 3D governing Eqs. (5)-(8) naturally split into the following five conjugated 2D problems

$$\frac{1}{\rho}\frac{\partial p^{(j)}}{\partial r} = \nu\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}ru_r^{(j)} + \frac{\partial^2 u_r^{(j)}}{\partial z^2} - \frac{\beta_j}{r^2}u_r^{(j)} - \frac{\delta_j}{r^2}u_\phi^{(j-(-1)^j)}\right], \qquad (16)$$

$$\frac{1}{\rho}\frac{\partial p^{(j)}}{\partial z} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_z^{(j)}}{\partial r} + \frac{\partial^2 u_z^{(j)}}{\partial z^2} - \frac{\beta_j}{r^2}u_z^{(j)}\right], \qquad (17)$$

$$\frac{\delta_j}{2}\frac{p^{(j-(-1)^j)}}{\rho r} = \nu\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}ru_\phi^{(j)} + \frac{\partial^2 u_\phi^{(j)}}{\partial z^2} - \frac{\beta_j}{r^2}u_\phi^{(j)} + \frac{\delta_j}{r^2}u_r^{(j-(-1)^j)}\right], \qquad (18)$$

$$\frac{1}{r}\frac{\partial}{\partial r}ru_r^{(j)} + \frac{\partial u_z^{(j)}}{\partial z} + \frac{\delta_j}{2r}u_\phi^{(j-(-1)^j)} = 0, \qquad (19)$$

where $j = 0,1,2,3,4$ and

$$\beta_j = \begin{cases} 0, & j=0 \\ 1, & j=1,2 \\ 4, & j=3,4 \end{cases}, \qquad \delta_j = \begin{cases} 0, & j=0 \\ (-1)^{j+1}2, & j=1,2 \\ (-1)^{j+1}4, & j=3,4 \end{cases}. \tag{20}$$

The boundary conditions for $u_r^{(j)}$, $u_\phi^{(j)}$ and $u_z^{(j)}$ are easily derived from Eqs. (9)-(15). Thus, we have reduced the original 3D problem to the five 2D problems, which significantly reduces the complexity of the problem.

System of the Eqs. (16)-(19) for $j = 0$ uncoupled from the other four systems

$$\frac{1}{\rho}\frac{\partial p^{(0)}}{\partial r} = \nu\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r u_r^{(0)} + \frac{\partial^2 u_r^{(0)}}{\partial z^2}\right], \tag{21}$$

$$\frac{1}{\rho}\frac{\partial p^{(0)}}{\partial z} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u_z^{(0)}}{\partial r} + \frac{\partial^2 u_z^{(0)}}{\partial z^2}\right], \tag{22}$$

$$0 = \nu\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r u_\phi^{(0)} + \frac{\partial^2 u_\phi^{(0)}}{\partial z^2}\right], \tag{23}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r} r u_r^{(0)} + \frac{\partial u_z^{(0)}}{\partial z}. \tag{24}$$

The boundary conditions for $u_r^{(0)}$, $u_\phi^{(0)}$ and $u_z^{(0)}$ are easily derived from Eqs. (9)-(15). They are

$$u_r|_\Gamma = -(G_{11} + G_{22})\frac{r}{2}, \tag{25}$$

$$u_\phi|_\Gamma = (G_{12} - G_{21})\frac{r}{2}, \tag{26}$$

$$u_z|_\Gamma = -(V^{(z)} + G_{33}z). \tag{27}$$

Component $u_\phi^{(0)}$ satisfies an elliptic equation and $u_\phi^{(0)} \equiv 0$ in the case $G_{12} = G_{21}$.

It is interesting to observe that systems for $j = 1$ and $j = 2$ are weakly coupled with each other, but uncoupled from the other systems. Thus we can solve these systems together by suitable numerical method. Systems of equations for $j = 3$ and $j = 4$ can be treated as systems for $j = 1$ and 2 provided the solution of the latter is already known.

4. SOLUTION OF BOUNDARY VALUE PROBLEM FOR BI-STOKESIAN EQUATION

The three Eqs. (21), (22) and (24) for $u_r^{(0)}$, $u_Z^{(0)}$ and $p^{(0)}$ are the 2D incompressible Stokes equations with radial symmetry (no dependence on the variable ϕ). Then it is possible to introduce stream function. The stream function can be defined to satisfy the continuity Eq. (24) for $u_r^{(0)}$ and $u_Z^{(0)}$

$$u_r^{(0)} = \frac{\partial \psi}{\partial z},\ u_z^{(0)} = -\frac{1}{r}\frac{\partial r\psi}{\partial r}. \tag{28}$$

Such introduced stream function satisfies to so called bi-Stokesian equation

$$E^4\psi = E^2(E^2\psi) = 0, \tag{29}$$

where E^2 is Stokesian operator define in cylindrical coordinates (4) as the following

$$E^2 \overset{\text{def}}{=} \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \frac{\partial^2}{\partial z^2}. \tag{30}$$

The bi-spherical coordinate system offers an important advantage in treating boundary value problem for bi-Stokesian equation, since both boundaries of rigid spheres can be represented as coordinate surfaces.

We introduce bi-spherical coordinates (ξ,ϕ',η) via their connection to cylindrical coordinates (r, ϕ, z) namely

$$r = \frac{a\sin\xi}{\cosh\eta - \cos\xi},\quad \phi' = \phi,\quad z = \frac{a\sinh\eta}{\cosh\eta - \cos\xi}, \tag{31}$$

Where the constant a is called focal distance. The coordinates (ξ,ϕ,η) vary in the intervals $[0, \pi]$, $[0, 2\pi]$ and $[\eta_1, \eta_2]$, respectively. The spheres' radii r_1 and r_2, and the distance of their centers from the origin d_1 and d_2 are computed by using the following relations

$$r_i = a\,\text{csch}\,|\eta_i|,\quad d_i = a\coth|\eta_i|. \tag{32}$$

The center to center distance between the spheres is $d= d_1 + d_2$. If r_1, r_2 and d are given, we can find a,η_1 and η_2 as follows

$$a = \frac{\sqrt{d^4 - 2d^2(r_1^2 + r_2^2) + (r_1^2 - r_2^2)^2}}{2d}, \tag{33}$$

$$\eta_1 = -\ln\Big(\frac{a}{r_1} + \sqrt{\frac{a^2}{r_1^2} + 1}\Big) = -\text{arcsinh}\frac{a}{r_1}, \tag{34}$$

$$\eta_2 = \ln\Big(\frac{a}{r_2} + \sqrt{\frac{a^2}{r_2^2} + 1}\Big) = \text{arcsinh}\frac{a}{r_2}. \tag{35}$$

The Stokesian operator (30) in bi-spherical coordinates has the following form

$$E^2 = \left(\frac{\cosh\eta - \cos\xi}{a}\right)^2\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + \frac{\cosh\eta - \cos\xi}{a}\left(\frac{\cos\xi\cosh\eta - 1}{a\sin\xi}\frac{\partial}{\partial\xi} - \frac{\sinh\eta}{a}\frac{\partial}{\partial\eta}\right) - \left(\frac{\cosh\eta - \cos\xi}{a\sin\xi}\right)^2. \tag{36}$$

To find solution of Eq. (29), let us consider related coupled system, i.e.

$$E^2\psi = \chi \ , \tag{37}$$

$$E^2\chi = 0. \tag{38}$$

It is well known (see [19]) that by means of the substitution

$$\chi = \sqrt{\cosh\eta - \cos\xi}\ \Phi, \tag{39}$$

Eq. (38) can be transformed into an equation with separating variables

$$\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\eta^2} + \cot\xi\frac{\partial\Phi}{\partial\xi} - \left(\frac{1}{4} + \frac{1}{\sin^2\xi}\right)\Phi = 0. \tag{40}$$

A solution for Φ is

$$\Phi(\eta,\mu) = \sum_{m=1}^{\infty}\left[L^{(m)}e^{(m+1/2)\eta} + K^{(m)}e^{-(m+1/2)\eta}\right]P_m^{(1)}(\mu), \tag{41}$$

where $\mu = \cos\xi$ and $P_m^{(1)}(\mu)$ denotes the associated Legendre polynomials (see for examples [17,20] and $L^{(m)}$, $K^{(m)}$ are arbitrary constants. In the same manner, we can change the dependent variable in Eq. (37)

$$\psi = \sqrt{\cosh\eta - \cos\xi}\ \varphi, \tag{42}$$

and to obtain equation with a separable left-hand side, namely

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \cot\xi\frac{\partial\varphi}{\partial\xi} - \left(\frac{1}{4} + \frac{1}{\sin^2\xi}\right)\varphi = \frac{a^2}{(\cosh\eta - \cos\xi)^{5/2}}\chi. \tag{43}$$

Substituting Eq. (41) into Eq. (43), yields

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \cot\xi\frac{\partial\varphi}{\partial\xi} - \left(\frac{1}{4} + \frac{1}{\sin^2\xi}\right)\varphi = \left(\frac{a}{\cosh\eta - \mu}\right)^2\sum_{m=1}^{\infty}\left[L^{(m)}e^{(m+1/2)\eta} + K^{(m)}e^{-(m+1/2)\eta}\right]P_m^{(1)}(\mu) \tag{44}$$

The key idea here is to make use of the generating function for Chebyshev polynomials of the second kind, $U_m(\mu)$ (see [20]), and express the right-hand side of Eq. (44) into series with respect to the associated Legendre polynomials. This is one of the main contributions of the present paper.

The generating function for Chebyshev polynomials of the second kind is defined as follow

$$G(t,\mu) = \frac{1}{1-2\mu t + t^2} = \sum_{m=0}^{\infty}t^m U_m(\mu);\ |t|<1,\ |\mu|\le 1.$$

For the derivative of the generating function, we get

$$\frac{\partial G}{\partial\mu} = \frac{2t}{(1-2\mu t+t^2)^2} = \sum_{m=1}^{\infty}t^m U_m'(\mu);\quad U_0'(\mu) = 0.$$

It follows that

$$\left(\frac{a}{\cosh\eta-\mu}\right)^2=2a^2\sum_{m=1}^{\infty}e^{-(m+1)|\eta|}U'_m(\mu).$$

At this stage, we need to derive a representation of the products of Chebyshev polynomials of the second kind with associated Legendre polynomials into series with respect to the associated Legendre polynomials. Formally, such a series can be written as

$$P_m^{(1)}(\mu)U'_l(\mu)=\sum_{k=1}^{\infty}p_k^{ml}P_k^{(1)}(\mu),\quad \text{where}\quad p_k^{ml}=0\quad \text{for}\quad k\geq m+l, \tag{45}$$

where

$$p_k^{ml}=\frac{2k+1}{2k(k+1)}\int_{-1}^{1}P_m^{(1)}(\mu)U'_l(\mu)P_k^{(1)}(\mu)d\mu.$$

Now Eq. (44) can be rewritten as follows

$$\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial\eta^2}+\cot\xi\frac{\partial\varphi}{\partial\xi}-\left(\frac{1}{4}+\frac{1}{\sin^2\xi}\right)\varphi=\sum_{k=1}^{\infty}Q_k(\eta)P_k^{(1)}(\mu), \tag{46}$$

where the following notation is adopted

$$Q_k(\eta)\overset{\text{def}}{=}2a^2\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}p_k^{ml}e^{-(l+1)|\eta|}\left(L^{(m)}e^{(m+1/2)\eta}+K^{(m)}e^{-(m+1/2)\eta}\right). \tag{47}$$

Here we can make use of the separability of the above Eq. (46) and seek the solution in the form of series with respect to associated Legendre polynomials

$$\varphi=\sum_{k=1}^{\infty}f_k(\eta)P_k^{(1)}(\mu). \tag{48}$$

After some tedious but straightforward computations, Eq. (46) adopts the following form

$$\sum_{k=1}^{\infty}f''_k(\eta)P_k^{(1)}(\mu)+(1-\mu^2)\sum_{k=1}^{\infty}f_k(\eta)P_k^{(1)\prime\prime}(\mu)-2\mu\sum_{k=1}^{\infty}f_k(\eta)P_k^{(1)\prime}(\mu)$$
$$-\left(\frac{1}{4}+\frac{1}{1-\mu^2}\right)\sum_{k=1}^{\infty}f_k(\eta)P_k^{(1)}(\mu)=\sum_{k=1}^{\infty}Q_k(\eta)P_k^{(1)}(\mu), \tag{49}$$

which breaks naturally into the two following independent equations

$$\sum_{k=1}^{\infty}\left((1-\mu^2)P_k^{(1)\prime\prime}(\mu)-2\mu P_k^{(1)\prime}(\mu)+\left[k(k+1)-\frac{1}{1-\mu^2}\right]P_k^{(1)}(\mu)\right)=0, \tag{50}$$

$$\sum_{k=1}^{\infty}\left[f''_k(\eta)-\left(k+\frac{1}{2}\right)^2f_k(\eta)\right]P_k^{(1)}(\mu)=\sum_{k=1}^{\infty}Q_k(\eta)P_k^{(1)}(\mu). \tag{51}$$

Therefore, we can find $f_k(\eta)$ by solving

$$f''_k(\eta)-\left(k+\frac{1}{2}\right)^2f_k(\eta)=Q_k(\eta). \tag{52}$$

The general solution of the homogeneous equation is

$$f_{k_c}(\eta) = C_k^{(1)} e^{(k+\frac{1}{2})\eta} + C_k^{(2)} e^{-(k+\frac{1}{2})\eta}. \quad (53)$$

where $C_k^{(1)}$ and $C_k^{(2)}$ are an arbitrary constants. A particular solution $f_{k_p}(\eta)$ of non-homogeneous Eq. (52) can be found by using the method of undetermined coefficients. The procedure is straightforward but tedious, and the resulting expressions for stream function are the followings

$$\psi(\eta,\mu) = (\cosh\eta - \mu)^{\frac{1}{2}} \sum_{k=1}^{\infty} Z_k(\eta) P_k^{(1)}(\mu) = (\cosh\eta - \mu)^{\frac{1}{2}} \sum_{k=1}^{\infty} [f_{k_c}(\eta) + f_{k_p}(\eta)] P_k^{(1)}(\mu), \quad (54)$$

Functions $f_{k_p}(\eta)$ are defined by the following

- If $\eta \le 0$, then $f_{k_p}(\eta) = \sum_{m=1}^{\infty} L^{(m)} \varepsilon_m^k(\eta) + \sum_{m=1}^{\infty} K^{(m)} \lambda_m^k(\eta)$.
- If $\eta \ge 0$, then $f_{k_p}(\eta) = \sum_{m=1}^{\infty} L^{(m)} \omega_m^k(\eta) + \sum_{m=1}^{\infty} K^{(m)} \tau_m^k(\eta)$.

where

$$\varepsilon_m^k(\eta) = \sum_{\substack{l=\max\{1,k-m+1\} \\ l \ne k-m-1}}^{\infty} \frac{2a^2 p_k^{ml} e^{(l+m+\frac{3}{2})\eta}}{\left(m+l+\frac{3}{2}\right)^2 - \left(k+\frac{1}{2}\right)^2} + \sum_{\substack{l=\max\{1,k-m+1\} \\ l=k-m-1}}^{\infty} \frac{a^2 p_k^{ml} \eta e^{(l+m+\frac{3}{2})\eta}}{m+l+\frac{3}{2}},$$

$$\lambda_m^k(\eta) = \sum_{\substack{l=\max\{1,k-m+1\} \\ l \ne m-k-1 \text{ or } l \ne k+m}}^{\infty} \frac{2a^2 p_k^{ml} e^{(l-m+\frac{1}{2})\eta}}{\left(l-m+\frac{1}{2}\right)^2 - \left(k+\frac{1}{2}\right)^2} + \sum_{\substack{l=\max\{1,k-m+1\} \\ l \ne m-k-1 \text{ or } l=k+m}}^{\infty} \frac{a^2 p_k^{ml} \eta e^{(l-m+\frac{1}{2})\eta}}{l-m+\frac{1}{2}},$$

$$\omega_m^k(\eta) = \sum_{\substack{l=\max\{1,k-m+1\} \\ l \ne m-k-1 \text{ or } l \ne k+m}}^{\infty} \frac{2a^2 p_k^{ml} e^{(m-l-\frac{1}{2})\eta}}{\left(m-l-\frac{1}{2}\right)^2 - \left(k+\frac{1}{2}\right)^2} + \sum_{\substack{l=\max\{1,k-m+1\} \\ l \ne m-k-1 \text{ or } l=k+m}}^{\infty} \frac{a^2 p_k^{ml} \eta e^{(m-l-\frac{1}{2})\eta}}{m-l-\frac{1}{2}},$$

$$\tau_m^k(\eta) = \sum_{\substack{l=\max\{1,k-m+1\} \\ l \ne k-m-1}}^{\infty} \frac{2a^2 p_k^{ml} e^{-(l+m+\frac{3}{2})\eta}}{\left(m+l+\frac{3}{2}\right)^2 - \left(k+\frac{1}{2}\right)^2} + \sum_{\substack{l=\max\{1,k-m+1\} \\ l=k-m-1}}^{\infty} \frac{a^2 p_k^{ml} \eta e^{-(l+m+\frac{3}{2})\eta}}{-\left(m+l+\frac{3}{2}\right)}.$$

The unknown coefficients $C_k^{(1)}$, $C_k^{(2)}$, $L^{(m)}$ and $K^{(m)}$ are to be determined from boundary conditions (25)-(27).

5. Expanding the Boundary Conditions into Series with Respect to Associated Legendre Polynomials

In the previous section, we have obtained solution (54) for the stream function. To determine the unknown coefficients $C_k^{(1)}$, $C_k^{(2)}$, $L^{(m)}$ and $K^{(m)}$, we have to expand the boundary conditions for the stream functions into series with respect to the associated Legendre polynomials. The boundary conditions for stream functions ψ are the following

$$\left.\frac{\partial \psi}{\partial \eta}\right|_\Gamma - \frac{\sinh\eta}{\cosh\eta - \cos\xi}\psi\Big|_\Gamma = -h u_\xi^{(0)}\Big|_\Gamma, \tag{55}$$

$$\left.\frac{\partial \psi}{\partial \xi}\right|_\Gamma + \frac{(\cos\xi\cosh\eta - 1)}{\sin\xi(\cosh\eta - \cos\xi)}\psi\Big|_\Gamma = h u_\eta^{(0)}\Big|_\Gamma, \tag{56}$$

where $u_\xi^{(0)}$ and $u_\eta^{(0)}$ are the components of velocity vector with respect to bi-spherical coordinates and $h = \dfrac{a}{cosh\eta - cos\xi}$. The relations between velocity components in cylindrical coordinates (4) and bi-polar coordinates (31) read

$$u_\xi^{(0)} = \frac{h}{a}(\cos\xi\cosh\eta - 1)u_r^{(0)} - \frac{h}{a}(\sin\xi\sinh\eta)u_z^{(0)}, \tag{57}$$

$$u_\eta^{(0)} = -\frac{h}{a}(\sin\xi\sinh\eta)u_r^{(0)} - \frac{h}{a}(\cos\xi\cosh\eta - 1)u_z^{(0)}. \tag{58}$$

Using boundary conditions (25)-(27) yields

$$u_\xi^{(0)}\Big|_\Gamma = \frac{h}{a}\sin\xi\sinh\eta_i V^{(z)} - \frac{h^2}{2a}(G_{11}+G_{22})(\cos\xi\cosh\eta_i - 1)\sin\xi + \frac{h^2}{a}G_{33}\sin\xi\sinh^2\eta_i, \quad i = 1,2, \tag{59}$$

$$u_\eta^{(0)}\Big|_\Gamma = \frac{h}{a}(\cos\xi\cosh\eta_i - 1)V^{(z)} + \frac{h^2}{2a}(G_{11}+G_{22})\sin^2\xi\sinh\eta_i + \frac{h^2}{a}G_{33}(\cos\xi\cosh\eta_i - 1)\sinh\eta_i, \quad i = 1,2. \tag{60}$$

The boundary conditions given by Eqs. (55) and (56) and Eqs. (59) and (60) require the representation of both sides in the form of series with respect to associated Legendre polynomials. Using generating function method [17,18] and the properties of associated Legendre polynomials [20], we can first compute the values of function $Z_k(\eta_i)$, $i = 1,2$ and its derivatives $Z_k'(\eta_i)$, $i = 1,2$ at the sphere boundaries. The following system of equations for $Z_k(\eta_i)$, $i = 1, 2$, $Z_k'(\eta_i)$, $i = 1,2$ are derived to satisfy boundary conditions (55) and (56), (59) and (60)

$$\left(\frac{1-k}{2k-1}\right)Z'_{k-1}(\eta_i)-\frac{\sinh\eta_i}{2}Z_k(\eta_i)+\cosh\eta_i Z'_k(\eta_i)-\left(\frac{k+2}{2k+3}\right)Z'_{k+1}(\eta_i)$$
$$=2^{3/2}aV^{(z)}\sinh\eta_i e^{-(k+\frac{1}{2})|\eta_i|}+\frac{(G_{11}+G_{22})\sqrt{2}a^2}{3}\left[(k+2)e^{-(k+\frac{3}{2})|\eta_i|}-(k-1)e^{-(k-\frac{1}{2})|\eta_i|}\right] \quad (61)$$
$$+sign[\eta_i]\frac{G_{33}a^2 2^{3/2}\sinh\eta_i}{3}(2k+1)e^{-(k+\frac{1}{2})|\eta_i|},\quad i=1,2,$$

$$-\frac{(k-2)(k-1)}{2(2k-1)}Z_{k-2}(\eta_i)+\cosh\eta_i\frac{(k-1)k}{2k-1}Z_{k-1}(\eta_i)-\frac{2k(k+1)}{(2k-1)(2k+3)}Z_k(\eta_i)$$
$$-\cosh\eta_i\frac{(k+1)(k+2)}{2k+3}Z_{k+1}(\eta_i)+\frac{(k+2)(k+3)}{2(2k+3)}Z_{k+2}(\eta_i)$$
$$=-a2^{\frac{3}{2}}V^{(z)}\left[\cosh\eta_i\left(\frac{k-1}{2k-1}e^{-(k-\frac{1}{2})|\eta_i|}+\frac{k+2}{2k+3}e^{-(k+\frac{3}{2})|\eta_i|}\right)-e^{-(k+\frac{1}{2})|\eta_i|}\right]$$
$$+\frac{(G_{11}+G_{22})a^2 2^{\frac{3}{2}}\sinh\eta_i}{3}\left[\frac{(k-2)(k-1)}{2k-1}e^{-(k-\frac{1}{2})|\eta_i|}-\frac{(k+2)(k+3)}{2k+3}e^{-(k+\frac{3}{2})|\eta_i|}\right] \quad (62)$$
$$+\frac{G_{33}a^2(2)^{\frac{3}{2}}\sinh\eta_i}{3}\left[(k+2)e^{-(k+3/2)|\eta_i|}-(k-1)e^{-(k-1/2)|\eta_i|}\right],\quad i=1,2.$$

Using definition of $Z_k(\eta)$ and values of $Z_k(\eta_i), Z_k'(\eta_i)$, the system of linear algebraic recursion formulas can be derived as the following

$$Z_k(\eta_1)=e^{(k+\frac{1}{2})\eta_1}C_k^{(1)}+e^{-(k+\frac{1}{2})\eta_1}C_k^{(2)}+\sum_{m=1}^{\infty}L^{(m)}\varepsilon_m^k(\eta_1)+\sum_{m=1}^{\infty}K^{(m)}\lambda_m^k(\eta_1), \quad (63)$$

$$Z_k(\eta_2)=e^{(k+\frac{1}{2})\eta_2}C_k^{(1)}+e^{-(k+\frac{1}{2})\eta_2}C_k^{(2)}+\sum_{m=1}^{\infty}L^{(m)}\omega_m^k(\eta_2)+\sum_{m=1}^{\infty}K^{(m)}\tau_m^k(\eta_2), \quad (64)$$

$$Z_k'(\eta_1)=(k+\frac{1}{2})e^{(k+\frac{1}{2})\eta_1}C_k^{(1)}-(k+\frac{1}{2})e^{-(k+\frac{1}{2})\eta_1}C_k^{(2)}+\sum_{m=1}^{\infty}L^{(m)}\varepsilon_m'^k(\eta_1)+\sum_{m=1}^{\infty}K^{(m)}\lambda_m'^k(\eta_1), \quad (65)$$

$$Z_k'(\eta_2)=(k+\frac{1}{2})e^{(k+\frac{1}{2})\eta_2}C_k^{(1)}-(k+\frac{1}{2})e^{-(k+\frac{1}{2})\eta_2}C_k^{(2)}+\sum_{m=1}^{\infty}L^{(m)}\omega_m'^k(\eta_2)+\sum_{m=1}^{\infty}K^{(m)}\tau_m'^k(\eta_2) \quad (66)$$

Because the coefficients $C_k^{(1)}$, $C_k^{(2)}$, $L^{(m)}$ and $K^{(m)}$ become small with large k, simultaneous solution of these recursion equations for the first $\hat{K}$ sets yields $4\hat{K}$ coefficients, thereby determine the stream function according to Eq. (29).

6. Conclusions

In this paper we study the problem of Stokes flow over two nonintersecting, unequal spheres, when the flow field at infinity subjects to constant velocity gradient. Under assumption an arbitrary velocity gradient at infinity, the problem is 3D and does not allow for analytical solution. It has shown that problem can be reduced to a five partially coupled 2D problems. First one admits general solution in form of series with respect to associated Legendre polynomials. We use the method of generality functions to expand the boundary conditions into Legendre series and to obtain a closed linear algebraic system for the coefficients.

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