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Contributed Paper

Almost Primal Ideals in Commutative Rings

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ABSTRACT

Let I be a proper ideal of a commutative ring R. An element $a \in R$ is called almost prime to I provided that $ra \in I - I^2$ (with $r \in R$) implies that $r \in I$. We denote by $A(I)$ the set of all elements of R that are not almost prime to I. I is called an almost primal ideal of R if the set $A(I) \cup I^2$ forms an ideal of R. In this paper we first provide some results on almost primal ideals. We also study the relations among the primal ideals, weakly primal ideals and almost primal ideals of R.

Keywords: almost prime ideal, primal ideal, weakly primal ideal, weakly prime ideal.

Throughout, R will be a commutative ring with identity. (However, in most places the existence of an identity plays no role.) By a proper ideal I of R we mean an ideal I with $I \neq R$. The concept of primal ideals in a commutative ring R introduced and studied in [1] (see also [2, 3]). Let I be an ideal of R. An element $a \in R$ is called prime (resp. weakly prime) to I if $ra \in I$ (resp. $0 \neq ra \in I$) (where $r \in R$) implies that $r \in I$. Denote by $S(I)$ (resp. $W(I)$) the set of elements of R that are not prime (resp. are not weakly prime) to I. A proper ideal I of R is said to be primal if $S(I)$ forms an ideal (so 0 is not necessarily primal); this ideal is always a prime ideal, called the adjoint ideal P of I. In this case we also say that I is a P-primal ideal of R. Note that if $r \in R$ and $a \in S(I)$, then clearly $ra \in S(I)$. So what we require for I being primal is that if a and b are not prime to I, then their difference is also not prime to a I. Also, a proper ideal I of R is called weakly primal if the set $P = W(I) \cup \{0\}$ forms an ideal; this ideal is always a weakly prime ideal by [4, Proposition 4]. In this case we also say that I is a P-weakly primal ideal. Of course, a weakly prime ideal P of R is a proper ideal P with the property that for $a, b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. If R is not an integral domain, then 0 is a 0-weakly primal ideal of R (by definition), so a weakly primal ideal need not be primal.

In their paper [5], Bhatwadekar and Sharma introduced the concept of almost prime (resp. n-almost prime) ideals, where a proper ideal I of R is said to be almost prime if $a, b \in R$ with $ab \in I - I^2$ (resp. $ab \in I - I^n$) implies $a \in I$ or $b \in I$. Here we define two new concept of ideals: almost primal and n-almost primal ideals.

Definition 1. Let I be an ideal of R, and consider an integer $n \geq 2$. An element $a \in R$ is called almost prime (resp. n-almost prime) to I if $ra \in I - I^2$ (resp. $ra \in I - I^n$ (with $r \in I$) implies that $r \in I$.

Example 2. Let $R = Z/24Z$ and consider the ideal $I = 6Z/24Z$ of R. Clearly $I^2 = 12Z/24Z$. It is straight forward to show that $5 + 24Z$ is almost prime to I.

Remarks 3. Let I be an ideal of R. Denote by $A(I)$ the set of all elements of R that are not almost prime to I, and by $A_n(I)$ the set of all elements of R that are not n-almost prime to I (so $A_2(I) = A(I)$). Then:

(1) Every element of I^2 (resp. I^n) is almost prime (resp. n-almost prime) to I.

(2) If an element $a \in R$ is prime to I, then it is n-almost prime to I for every $n \geq 2$, but not conversely.

Lemma 4. Let I be an ideal of R. If $P = A(I) \cup I^2$ (resp. $P = A(I) \cup I^n$) is an ideal of R, then it is an almost prime (resp. n-almost prime) ideal of R.

Proof. Let $a,b \in R$ be such that $ab \in P\text{-}P^2$. Then, $rab \in I - I^2$ for some $r \in R - I$. Assume that $a \notin P$. Then a is almost prime to I. So $rb \in I - I^2$, implies that b is not almost prime to I, that is $b \in P$.

Definition 5. Let I be an ideal of R.

(1) I is called almost primal if the set $P = A(I) \cup I^2$ forms an ideal of R. This ideal P is an almost prime ideal of R, called the almost prime adjoint ideal of I. In this case we also say that I is a P-almost primal ideal.

(2) I is called n-almost primal if the set $P = A(I) \cup I^n$ forms an ideal of R. This ideal P is an n-almost prime ideal of R, called the n-almost prime adjoint ideal of I. In this case we also say that I is a $P-n$-almost primal ideal.

Note that a 2-almost primal ideal is just an almost primal ideal.

The following example shows that all ideals of R need not necessarily be almost primal.

Example 6. Let $R = Z/24Z$ and consider the ideal $I = 6Z/24Z$ of R. Clearly $I^2 = 12Z/24Z$. Then, $(2 + 24Z)(3 + 24Z) \in I - I^2$ with $2 + 24Z$, $3 + 24Z \in R - I$. So $2 + 24Z$ and $3 + 24Z$ are not almost prime to I. But $(2 + 24Z)+(3 + 24Z) = 5 + 24Z$ is almost prime to I. This shows that $A(I) \cup I^2$ is not an ideal of R. Therefore I is not almost primal.

Here we give several characterizations of almost primal ideals.

Proposition 7. Let I and P be proper ideals of R. For every integer $n \geq 2$, the following statements are equivalent:

(1) I is P-n-almost primal.

(2) For every $x \notin P - I^n$, $(I :_R x) = I \cup (I :_R x)$; and for every $x \in P - I^n$, $(I :_R x) \supset I \cup (I^n :_R x)$.

(3) for every $x \notin P - I^n$, $(I :_R x) = I$ or $(I :_R x) = (I^n :_R x)$; and for every $x \in P - I^n$, $(I :_R x) \supset I$ and $(I :_R x) \supset (I^n :_R x)$.

Proof. 1⇒2) Assume that I is P–n-almost primal. Then $P - I^n$ consists entirely of elements of R that are not n-almost prime to I. Let $x \notin P - I^n$. Then x is n-almost prime to I. Clearly $(I :_R x) \supseteq I \cup (I^n :_R x)$. For every $r \in (I :_R x)$, if $rx \in I^n$, then $r \in (I^n :_R x)$, and if $rx \notin I^n$, then x n-almost prime to I gives $r \in I$. Hence $r \in I \cup (I^n :_R x)$, that is $(I :_R x) \subseteq I \cup (I^n :_R x)$. Therefore $(I :_R x) = I \cup (I^n :_R x)$.

Now assume that $x \in P - I^n$. Then x is not n-almost prime to I. So there exists $r \in R - I$ such that $rx \in I - I^n$. Hence $r \in (I :_R x) - (I \cup (I^n :_R x))$.

2⇒3) Let $x \notin P - I^n$. Since $(I :_R x)$ is an ideal of R and $(I :_R x) = I \cup (I^n :_R x)$, either $I \subseteq (I^n :_R x)$ or $(I^n :_R x) \subseteq I$. So either $(I :_R x) = I$ or $(I :_R x) = (I^n :_R x)$. Moreover, for every $x \in P - I^n$, $(I :_R x) \supset I \cup (I^n :_R x)$. Hence $(I :_R x) \supset I$ and $(I :_R x) \supset (I^n :_R x)$.

3⇒1) By (3), $P - I^n$ consists exactly of all elements of R that are not n-almost prime to I. Hence I is $P-n$-almost primal.

Theorem 8. Let $n > 2$. Let P be an n-almost prime ideal of R. If P is not prime, then $P^2 = P^n$.

Proof. Suppose that $P^2 \neq P^n$. We show that P is prime. Assume that $a,b \in R$ are such that $ab \in P$. We have two cases $ab \notin P^n$ and $ab \in P^n$. In the former case, P n-almost prime gives $a \in P$ or $b \in P$. So we can assume that the latter case holds. First suppose that $aP \not\subset P^n$. Then $aP_0 \notin P^n$ for some $P_0 \in P$. In this case $a(b + P_0) \in P - P^n$, and P n-almost prime gives $a \in P$ or $b + P_0 \in P$. Hence $a \in P$ or $b \in P$. Thus we may assume that $aP \subseteq P^n$.

Similarly we can assume that $aP \subseteq P^n$. There exist $p,q \in P$ with $pq \notin P^n$ since $P^2 \neq P^n$. In this case $(a+p)(b+q) \in P - P^n$. So P n-almost prime gives $a+p \in P$ or $b+p \in P$. Therefore $a \in P$ or $b \in P$. Hence P is prime.

Let I be a proper ideal of R and let $n>2$. Then $I - I^2 \subseteq A(I)$ (resp. $I - I^n \subseteq A_n(I)$). Hence, if I is a P-almost primal (resp. $P-n$-almost primal) ideal of R, then $I \subseteq P$. This fact is used in the proof of the following result.

Theorem 9. Let $n>2$. Let I be a $P-n$-almost primal ideal of the commutative ring R such that P is a prime ideal. If $I^2 \neq I^n$, then I is primal.

Proof. It is enough to show that $P = S(I)$. For every $a \in P$, if $a \in I^n$, then $a \in S(I)$ since $I \subseteq S(I)$; and if $a \notin I^n$, then a is not n-almost prime to I; so, by Remark 3, a is not prime to I, that is $P \subseteq S(I)$. For the reverse containment assume that $a \in S(I)$. There exists $r \in R - I$ with $ra \in I$. If $ra \notin I^n$, then a is not n-almost prime to I and so $a \in P$. Suppose that $ra \in I^n$. If $aI \not\subset I^n$, there exists $r_0 \in I$ such that $ar_0 \notin I^n$. Now $a(r+r_0) \in I - I^n$ with $r+r_0 \in R - I$ gives $a \in P$. So assume that $aI \subseteq I^n$. If $rI \not\subset I^n$, there exists $c \in I$ with $rc \notin I^n$. Hence from $(a+c)r \in I - I^n$ with $r \in R - I$ we get $a+c \in P$. Further, $c \notin I^n$, for otherwise, $rc \in I^n$, a contradiction. Since $I - I^n \subseteq P$, we have $c \in P$. Consequently, $a \in P$. Thus assume that $rI \subseteq I^n$. Since $I^2 \neq I^n$, there exist $a_0, b_0 \in I$ with $a_0 b_0 \notin I^n$. Now $(a+a_0)(r+b_0) \in I - I^n$ with $r+b_0 \in R - I$ implies that $a+a_0 \in P$. Hence $a \in P$ again by the previous argument. Therefore in any case a lies in P, that is $S(I) \subseteq P$, and hence $P = S(I)$. So I is $P-$primal.

Definition 10. Let I be a proper ideal of R. I is said to be almost primary if, for $a,b \in R$, $ab \in I - I^2$ implies $a \in I$ or $b \in \sqrt{I}$. If $\sqrt{I} = P$, then I is called $P-$almost primary.

Theorem 11. Every almost primary ideal of a commutative ring R is almost primal. In particular, every almost prime ideal is almost primal.

Proof. Let I be a $P-$almost primary ideal of R. For every $a \in A(I)$, there exists $r \in R - I$ with $ra \in I - I^2$. Then as I is $P-$ almost primary we get $a \in \sqrt{I} = P$. This implies that $A(I) \cup I^2 \subseteq P$. Now assume that $a \in P - I^2$. Suppose that m is the least positive integer for which $a^m \in I - I^2$. Then $aa^{m-1} \in I - I^2$ with $a^{m-1} \in R - I$ implies that a is not almost prime to I. Thus $P \subseteq A(I) \cup I^2$, and hence we have $P = A(I) \cup I^2$, that is I is $P-$almost primal. The proof of the last part is easy, because every almost prime ideal is almost primary.

The concept of weakly primary ideals introduced in [6]. Recall that a proper ideal Q of R is said to be weakly primary if $0 \neq ab \in Q$ implies $a \in Q$ or $b \in \sqrt{Q}$. In this case $P = \sqrt{Q}$ is a weakly prime ideal of R, and Q is called $P-$weakly primary. The following result provides the relations between almost primal and weakly primal ideals as well as between almost primary and weakly primary ideals.

Theorem 12. Let I and P be proper ideals of R with $I \subseteq P$. Then

(1) I is a $P-$almost primal ideal of R if and only if I / I^2 is a P / I^2-weakly primal ideal of R / I^2.

(2) I is a $P-$almost primary ideal of R if and only if I / I^2 is a P / I^2 $-$weakly primary ideal of R / I^2.

Proof. Set $\overline{R} = R / I^2, \overline{I} = I / I^2, \overline{P} = P / I^2$ and denote by $\overline{a}$ the coset a / I^2 for each $a \in R$.

(1) First assume that I is a $P-$almost primal ideal of R. Suppose that $\overline{a} \in \overline{R}$ is not weakly prime to $\overline{I}$. Then $\overline{a} \neq \overline{0}$, and $\overline{0} \neq \overline{a}\overline{b} \in \overline{I}$ for some $\overline{b} \in \overline{R} - \overline{I}$. This implies that $a \notin I^2$ and $ab \in I - I^2$ with $b \in R - I$, that is a is not almost prime to I. So that $a \in P - I^2$, that is $\overline{a} \in \overline{P} - \{\overline{0}\}$. Therefore $w(\overline{I}) \cup \{\overline{0}\} \subseteq \overline{P}$, where $w(\overline{I})$ denotes the set of all $\overline{a} \in \overline{R}$ such that $\overline{a}$ is not weakly prime to $\overline{I}$. Now pick an element $\overline{c} \in \overline{P} - \{\overline{0}\}$. Then there exists $a \in I^2$ such that $c + a \in P$. This implies that $c + a$ is not almost prime to I. So $(c + a)d \in I - I^2$ for some $d \in R - I$. Since $(c + a) - c = a \in I^2$, we have $\overline{c+a} = \overline{c}$. Then $\overline{0} \neq \overline{c}\overline{d} \in \overline{I}$ with $\overline{d} \in \overline{R} - \overline{I}$. Therefore $\overline{c}$ is not weakly prime to $\overline{I}$, that is $\overline{P} \subseteq w(\overline{I}) \cup \{\overline{0}\}$, and hence we have $\overline{P} = w(\overline{I}) \cup \{\overline{0}\}$. Consequently $\overline{I}$ is a $\overline{P}-$weakly primal ideal of $\overline{R}$. Conversely, if $\overline{I}$ is a $\overline{P}-$weakly primal ideal of R, it is straight forward as above to show that $A(I) = P - I^2$. This implies that I is a $P-$almost primal ideal of R.

(2) Assume that I is $P-$almost primary in R. Let $\overline{a}\overline{b}, \in \overline{R}$ be such that $0 \neq \overline{a}\overline{b} \in \overline{I}$. Then $ab \in I - I^2$ and I almost primary gives either $a \in I$ or $b \in \sqrt{I} = P$. Therefore, either $\overline{a} \in \overline{I}$ or $\overline{b} \in \overline{P}$. Hence $\overline{I}$ is weakly primary. Conversely, assume that $\overline{I}$ is $P-$weakly primary. Let $a,b \in R$ be such that $ab \in I - I^2$. Then $0 \neq \overline{a}\overline{b} \in \overline{I}$. Since $\overline{I}$ is $\overline{P}-$weakly primary, we get either $\overline{a} \in \overline{I}$ or $\overline{b} \in \sqrt{\overline{I}} = \overline{P}$. So either $a \in I$ or $b \in P$ as needed.

Proposition 13. Let J and P be ideals of R with $J \subseteq P^2$. Then P is an almost prime ideal of R if and only if P / J is an almost prime ideal of R / J.

Proof. If P is almost prime, then P / J is almost prime by [7, Proposition 15]. Now assume that P / J is almost prime, and let $a,b \in R$ are such that $ab \in P - P^2$.

Then $(a + J)(b+) \in (P / J) - (P / J)^2$ since $J \subseteq P^2$. As P / J is almost prime, it follows that either $a + J$ or $b + J$ is in P / J. Thus either a or b is in P, that is P is almost prime.

Theorem 14. Let I and J be ideals of R with $J \subseteq I^i \subseteq P$ for $i = 1,2$. Then I is an almost primal ideal of R if and only if I / J is an almost primal ideal of R / J.

Proof. First assume that I is a $P-$almost primal ideal of R. Then, by Lemma 4, and [7, Proposition 15], P / J is an almost prime ideal of R / J. We show that I / J is a $P / J-$almost primal ideal of R / J. We claim that $R / J = A(I / J) \cup (I / J)^2$. Let $A + J \in P / J - (I / J)^2$. Then $a \in P - I^2$, that is a is not almost prime to I. Hence there exists $r \in R - I$ with $ra \in I - I^2$. So we have $(r + J)(a + J) \in (I / J) - (I / J)^2$ with $r + J \in R / J - I / J$. Hence $a + J$ is not almost prime to I / J. Now assume that $b + J$ is not almost prime to I / J. Then $b + J \notin (I / J)^2$ and there exists $r + J \in R / J - I / J$ with $rb + J = (r + J)(b + J) \in I / J - (I / J)^2$. Hence $rb \in I - I^2$ with $r \notin I$, that is b is not almost prime to I. Hence $b \in P - I^2$ and so b

$+ J \in P/J - (I/J)^2$. We have already shown that $P/J - (I/J)^2$ consists exactly of elements of R/J that are not almost prime to I/J. Hence I/J is almost primal with the adjoint ideal P/J. Conversely, assume that I/J is a P/J–almost primal ideal of R/J. We will show that I is a P–almost primal ideal of R. By Lemma 4, P/J is an almost prime ideal of R/J. Also $J \subseteq I^2 \subseteq P^2$ by our assumption. Hence P is an almost prime ideal of R by Proposition 13. It suffices to show that P–I^2 consists exactly of elements of R that are not almost prime to I. Let $a \in P - I^2$. Then $a + J \in (P/J) - (I/J)^2$, that is $a + J$ is not almost prime to I/J. Hence there exists $r + J \in (R/J) - (I/J)$ such that $(a + J)(r + J) \in (I/J) - (I/J)^2$. therefore $ar \in I - I^2$ with $r \in R - J$. This implies that a is not almost prime to I. Now assume that b is not almost prime to I. Then there exists $r \in R - I$ with $rb \in I - I^2$. Consequently $(r + J)(b + J) = rb + J \in (I/J) - (I/J)^2$ with $r + J \notin I/J$. Hence $b + J$ is not almost prime to I/J, that is $a + J \in (P/J) - (I/J)^2$; so $a \in P - I^2$. It follows that $P - I^2$ is exactly the set of elements of R that are not almost prime to I. Hence I is P–almost primal.

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