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Contributed Paper

New Results on Parallel Alternating Iterative Methods

Guangbin Wang*[a], Ting Wang [a], Fuping Tan [b] and Shuqian Shen [c]
[a] Department of Mathematics, Qingdao University of Science and Technology, China.
[b] Department of Mathematics, Shanghai University, China.
[c] School of Science, China University of Petroleum, China.
*Author for correspondence; e-mail: wguangbin750828@sina.com

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ABSTRACT

In this paper, we give some new results on parallel synchronous alternating iterative methods for solving the system of linear equations $Ax = b$. Firstly, we give two convergence theorems when the coefficient matrix is a nonsingular H-matrix by using an H-compatible multisplitting. Secondly, we give some comparison theorems of the spectral radius of the iterative matrix when the coefficient matrix is a monotone matrix and give two examples to show our results.

Keywords: parallel synchronous alternating method, H-matrix, H-compatible multisplitting, monotone matrix, weak nonnegative of the first type, weak nonnegative of the second type

1. INTRODUCTION

For the large system of linear equations

$$Ax = b \tag{1.1}$$

where A is a nonsingular square matrix of order n, $x, b \in R^n$. Benzi and Szyld [1] analyzed the following alternating method: Given an initial vector $x^{(0)}$, for $k = 0,1,2,...$,

$$x^{(k+\frac{1}{2})} = M^{-1}Nx^{(k)} + M^{-1}b,$$
$$x^{(k+1)} = P^{-1}Qx^{(k+\frac{1}{2})} + P^{-1}b,$$

where $A = M - N = P - Q$ are two splittings of A.

In paper [2], Climent *et al.* introduced two parallel synchronous alternating iterative methods.

Method 1: Let

$$(M_{j,l}, N_{j,l}, E_{j,l})_{l=1}^{p}, j = 1,2,\cdots,q, \tag{1}$$

be q multisplittings of A. For the q multisplittings, let

$$x^{k+j/q} = \sum_{l=1}^{p} E_{j,l}(M_{j,l}^{-1}N_{j,l})^{\mu(j,k,l)} x^{(k+(j-1)/q)} + \sum_{l=1}^{p} E_{j,l}(\sum_{i=0}^{\mu(j,k,l)-1} (M_{j,l}^{-1}N_{j,l})^{i}) M_{j,l}^{-1}b$$

where $\mu(j, k, l)$ denotes the number of local iterations in the l th processor at the k th global iteration using the j th multisplitting.

We can eliminate vectors $x^{k+j/q}$ for $j = 1,2,..., q - 1$, obtaining the single iterative process

$$x^{(k+1)} = \prod_{j=1}^{q} (\sum_{l=1}^{p} E_{q-j+1,l} (M_{q-j+1,l}^{-1} N_{q-j+1,l})^{\mu(q-j+1,k,l)}) x^k \tag{2}$$

$$+\sum_{m=1}^{q}[\prod_{j=1}^{q-m}(\sum_{l=1}^{p}E_{q-j+1,l}(M_{q-j+1,l}^{-1}N_{q-j+1,l})^{\mu(q-j+1,k,l)})]$$
$$\times[\sum_{l=1}^{p}E_{m,l}(\sum_{i=0}^{\mu(m,k,l)-1}(M_{m,l}^{-1}N_{m,l})^{i})M_{m,l}^{-1}]b,$$

for k = 0,1,2,.... Where we assume that

$$\prod_{j=1}^{0}(\sum_{l=1}^{p}E_{q-j+1,l}(M_{q-j+1,l}^{-1}N_{q-j+1,l})^{\mu(q-j+1,k,l)})=I.$$

The iterative matrix is

$$S^{(k)}=\prod_{j=1}^{q}(\sum_{l=1}^{p}E_{q-j+1,l}(M_{q-j+1,l}^{-1}N_{q-j+1,l})^{\mu(q-j+1,k,l)}). \quad (3)$$

Method 2: Let

$$(M_{j,l},N_{j,l},E_l)_{l=1}^{p},\ j=1,2,\cdots,q, \quad (4)$$

be q multisplittings of A, each one with the same weighting matrices $\{E_l\}_{l=1}^{p}$. The single iterative process

$$x^{(k+1)}=\sum_{l=1}^{p}E_lT_l^{(k)}x^{(k)}+\sum_{l=1}^{p}E_lR_l^{(k)}b,$$
$$k=0,1,2,\cdots,$$

whose iterative matrix is

$$T^{(k)}=\sum_{l=1}^{p}E_lT_l^{(k)}$$
$$=\sum_{l=1}^{p}E_l\prod_{j=1}^{q}(M_{q-j+1,l}^{-1}N_{q-j+1,l})^{\mu(q-j+1,k,l)} \quad (5)$$
$$k=0,1,2,\cdots,$$

where

$$T_l^{(k)}=\prod_{j=1}^{q}(M_{q-j+1,l}^{-1}N_{q-j+1,l})^{\mu(q-j+1,k,l)}, \quad (6)$$

$$R_l^{(k)}=\sum_{i=1}^{q}[\prod_{j=1}^{q-i}(M_{q-j+1,l}^{-1}N_{q-j+1,l})^{\mu(q-j+1,k,l)}]$$
$$\times[(\sum_{i=0}^{\mu(i,k,l)-1}(M_{i,l}^{-1}N_{i,l})^{m})M_{i,l}^{-1}].$$

In paper [2], Climent *et al.* gave convergence theorems for the parallel synchronous alternating methods when the coefficient matrix is a monotone matrix by using the weak nonnegative multisplitting of the first or of the second type. Also, they gave convergence theorems for these methods when the coefficient matrix is a symmetric positive definite matrix by using the P-regular multisplitting.

In this paper, we give two convergence theorems for the parallel synchronous alternating methods when the coefficient matrix is a nonsingular H-matrix by using the H-compatible multisplitting. Also, we give some comparison theorems of the spectral radius of the iterative matrix when the coefficient matrix is a monotone matrix by using the weak nonnegative multisplitting of different type.

2. PRELIMINARIES

Let $A \in R^{n\times n}$. We denote by $A \geq 0$ a nonnegative matrix, $|A|$ the absolute value of matrix A, $\rho(A)$ the spectral radius of A, and $<A>$ the comparison matrix of A.

Definition 2.1[3] A nonsingular matrix A is called an M-matrix if A can be expressed as $A=sI-B$, with $B \geq 0$, $s>0$, and $\rho(B)<s$. A nonsingular matrix A is said to be an H-matrix if $<A>$ is an M-matrix. A is called a monotone matrix if $A^{-1} \geq 0$.

Definition 2.2[4] Let $A \in R^{n\times n}$. $A = M-N$ ($M, N \in R^{n\times n}$) is called an H-splitting if $<M>-|N|$ is an M-matrix. If $<A>=<M>-|N|$, then $A = M - N$ is called an H-compatible splitting.

Definition 2.3[5,6] Let $A = B - C$ be a splitting of A. If $B^{-1} \geq 0$, $B^{-1} \geq C \geq 0$, then $A = B - C$ is a weak nonnegative splitting of the first type. If $B^{-1} \geq 0$, $CB^{-1} \geq 0$, then $A = B - C$ is a weak nonnegative splitting of the second type.

If $B^{-1} \geq 0$, $B^{-1}C \geq 0$, $CB^{-1} \geq 0$, then $A=B-C$ is a nonnegative splitting. If B is an M-matrix and $C \geq 0$, then $A=B-C$ is an

M-splitting.

Obviously, an M-splitting is a weak nonnegative splitting of the first type.

Definition 2.4[7] We say that the set of splittings $\{(M_l, N_l, E_l)\}_{l=1}^{p}$ is a multisplitting of A if

- $A=M_l - N_l$, for $l = 1, 2,..., p$ is a splitting,
- $E_l \geq 0$, for $l = 1, 2,..., p$ are diagonal matrices called weighting matrices,
- $\sum_{l=1}^{p} E_l = I$, where I is the identity matrix.

As a generalization of Definition 2.2 and Definition 2.3, we say that a multisplitting is H-compatible, weak nonnegative of the first type, weak nonnegative of the second type, nonnegative, M-multisplitting, respectively, if each splitting of the multisplitting is H-compatible, weak nonnegative of the first type, weak nonnegative of the second type, nonnegative, M-splitting, respectively.

Lemma 2.1[8] Let $A\in R^{n\times n}$. If A is a nonsingular M-matrix, $B\in Z^{n\times n}$ and $A\leq B$, then B is a nonsingular M-matrix.

Lemma 2.2[9] If $A\in R^{n\times n}$ is a nonsingular H-matrix, then $|A^{-1}|\leq\langle A\rangle^{-1}$.

3. Convergence Theorems

In this section, we give two convergence theorems for the parallel synchronous alternating methods when the coefficient matrix is a nonsingular H-matrix by using the H-compatible multisplitting.

Lemma 3.1[2] Let A be a nonsingular matrix. Assume that A is monotone and that multisplittings are weak nonnegative of the first type. If

$$\mu(j,k,l)\geq 1,\ j=1,2,\cdots,q;\quad l=1,2,\cdots,p;\ k=1,2,\cdots, \tag{7}$$

then Method 1 converges to the unique solution of system (1) for any initial vector $x^{(0)}$.

Theorem 3.1 Let $A\in R^{n\times n}$ be a nonsingular H-matrix, and

$A = M_{jl}- N_{jl}$ (j - 1, 2,..., q)

are H-compatible multisplittings. If

$$\mu(j,k,l)\geq 1,\ j=1,2,\cdots,q,\quad l=1,2,\cdots,p,\ k=1,2,\cdots, \tag{8}$$

then Method 1 converges to the unique solution of (1.1) for any starting vector $x^{(0)}$.

Proof: In order to prove that Method 1 is convergent, we only show that

$$\lim_{k\to\infty}(S^{(k)}S^{(k-1)}\cdots S^{(1)})=0.$$

From $A = M_{q-j+1,l}- N_{q-j+1,l}$ (j = 1, 2,..., q) are H-compatible multisplittings, we have $\langle A\rangle=\langle M_{q-j+1,l}\rangle-|N_{q-j+1,l}|$ is a nonsingular M-matrix. From Lemma 2.1 we know that $\langle M_{q-j+1,l}\rangle$($j$ = 1, 2,..., q) are nonsingular M-matrices, thus

$$\langle A\rangle=\langle M_{q-j+1,l}\rangle-|N_{q-j+1,l}|$$

are M-multisplittings of $\langle A\rangle$(j =1,2,..., q) and $M_{q-j+1,l}$ (j =1,2,..., q) are nonsingular H-matrices. From Lemma 2.2 we know that

$$|M_{q-j+1,l}^{-1}|\leq\langle M_{q-j+1,l}\rangle^{-1}.$$

Thus we obtain

$$\begin{aligned}|S^{(k)}|&=|\prod_{j=1}^{q}\sum_{l=1}^{p}E_{q-j+1,l}(M_{q-j+1,l}^{-1}N_{q-j+1,l})^{\mu(q-j+1,k,l)}|\\ &\leq\prod_{j=1}^{q}\sum_{l=1}^{p}E_{q-j+1,l}(|M_{q-j+1,l}^{-1}||N_{q-j+1,l}|)^{\mu(q-j+1,k,l)}\\ &\leq\prod_{j=1}^{q}\sum_{l=1}^{p}E_{q-j+1,l}(\langle M_{q-j+1,l}\rangle^{-1}|N_{q-j+1,l}|)^{\mu(q-j+1,k,l)}\\ &=\overline{S}^{(k)}.\end{aligned}$$

So

$$|S^{(k-1)}|\leq\overline{S}^{(k-1)},\cdots,|S^{(1)}|\leq\overline{S}^{(1)}$$

and $0\leq|S^{(k)}||S^{(k-1)}|\cdots|S^{(1)}|$
$\leq\overline{S}^{(k)}\overline{S}^{(k-1)}\cdots\overline{S}^{(1)}$.

We use Lemma 3.1 to see immediately that

$$\lim_{k\to\infty}(\overline{S}^{(k)}\overline{S}^{(k-1)}\cdots\overline{S}^{(1)})=0.$$

Therefore,

$$\lim_{k\to\infty}(|S^{(k)}||S^{(k-1)}|\cdots|S^{(1)}|)=0.$$

Since

$0 \leq | S^{(k)} S^{(k-1)} \cdots S^{(1)} | \leq | S^{(k)} || S^{(k-1)} | \cdots | S^{(1)} |,$

we have

$$\lim_{k\to\infty}(| S^{(k)} S^{(k-1)} \cdots S^{(1)} |) = 0$$

and

$$\lim_{k\to\infty}(S^{(k)} S^{(k-1)} \cdots S^{(1)}) = 0,$$

thus we obtain the conclusion of this theorem.

Lemma 3.2[2] Let A be a nonsingular matrix. Assume that A is monotone and that the q multisplittings are weak nonnegative of the first type. If

$$\mu(j,k,l) \geq 1,\ j = 1,2,\cdots,q, \quad l = 1,2,\cdots,p,\ k = 1,2,\cdots, \tag{9}$$

then Method 2 converges to the unique solution of system (1) for any initial vector $x^{(0)}$.

Similar to the proof of Theorem 3.1, we can obtain the following theorem by using Lemma 3.2.

Theorem 3.2 Let $A \in R^{n\times n}$ be a nonsingular H-matrix, and

$$A = M_{jl} - N_{jl} (j = 1, 2, ..., q)$$

are H-compatible multisplittings. If

$$\mu(j,k,l) \geq 1,\ j = 1,2,\cdots, \quad q, l = 1,2,\cdots,p,\ k = 1,2,\cdots, \tag{10}$$

then Method 2 converges to the unique solution of (1.1) for any starting vector $x^{(0)}$.

4. Comparison Theorems

In this section, we give some comparison theorems of the spectral radius of the iterative matrices for the stationary parallel synchronous alternating methods when the coefficient matrix is a monotone matrix by using the weak nonnegative multisplittings of different type.

Lemma 4.1[10] Let $A^{-1} \geq 0$ and $A=M-N$ be a weak nonnegative splitting of the first type(respectively, second type). Then there exists a unique splitting $A=M-N$ associated with $T=(M^{-1}N)^p$, where $p \geq 1$, that is also weak nonnegative splitting of the first type (respectively, second type).

Lemma 4.2[11] Let $A \in R^{n\times n}$ and $A=M-N$ be a splitting. Then

$$M^{-1}NA^{-1} = A^{-1}NM^{-1}.$$

Lemma 4.3[12] Let $A \in R^{n\times n}$ be nonsingular. Let $T \in R^{n\times n}$ such that $I-T$ is nonsingular, then there exists a unique pair of matrices B and C, such that B is nonsingular, $T=B^{-1}C$ and $A=B-C$. Where $B=A(I-T)^{-1}$ and $C=B-A$.

Lemma 4.4 Let $A^{-1} \geq 0$. If

$$A=M_l-N_l = P_l-Q_l,\ M_l^{-1} \geq P_l^{-1} (l = 1, 2),..., p),$$

then

$$B_l^{-1} \geq D_l^{-1} (l = 1, 2),..., p),$$

where

$$B_l = A(I - (M_l^{-1}N_l)^{\mu(l)})^{-1},$$
$$D_l = A(I - (P_l^{-1}Q_l)^{\mu(l)})^{-1},$$

Proof: Since

$$P_l^{-1}Q_lA^{-1} = A^{-1} - P_l^{-1} \geq A^{-1} - M_l^{-1} = M_l^{-1}N_lA^{-1},$$

from Lemma 4.2, we have that

$$\begin{aligned}
(P_l^{-1}Q_l)^{\mu(l)}A^{-1} &= P_l^{-1}Q_lA^{-1}(Q_lP_l^{-1})^{\mu(l)-1} \\
&\geq M_l^{-1}N_lA^{-1}(Q_lP_l^{-1})^{\mu(l)-1} \\
&= M_l^{-1}N_lA^{-1}Q_lP_l^{-1}(Q_lP_l^{-1})^{\mu(l)-2} \\
&= M_l^{-1}N_lP_l^{-1}Q_lA^{-1}(Q_lP_l^{-1})^{\mu(l)-2} \\
&\geq M_l^{-1}N_lM_l^{-1}N_lA^{-1}(Q_lP_l^{-1})^{\mu(l)-2} \\
&= (M_l^{-1}N_l)^2A^{-1}(Q_lP_l^{-1})^{\mu(l)-2} \\
&\geq \cdots \geq (M_l^{-1}N_l)^{\mu(l)}A^{-1}.
\end{aligned}$$

From Lemma 4.3 we have that

$$\begin{aligned}
&B_l^{-1} - D_l^{-1} \\
&= (I - (M_l^{-1}N_l)^{\mu(l)})A^{-1} - (I - (P_l^{-1}Q_l)^{\mu(l)})A^{-1} \\
&= (P_l^{-1}Q_l)^{\mu(l)}A^{-1} - (M_l^{-1}N_l)^{\mu(l)}A^{-1} \\
&\geq 0.
\end{aligned}$$

Lemma 4.5[4] Let $A^{-1} \geq 0$ and

$$A = M_1 - N_1 = M_2 - N_2$$

be two weak nonnegative splittings of different type. If

$$M_1^{-1} \geq M_2^{-1},$$

then

$$\rho(M_1^{-1} N_1) \leq \rho(M_2^{-1} N_2) < 1.$$

4.1 Comparison theorems for the stationary version of Method 1

Lemma 4.6[2] Let A be a nonsingular matrix. Assume that

$$\mu(j, k, l) = \mu(j, l) \geq 1$$

for $k = 0, 1, 2, \cdots$.

(i) If A is monotone and the q multi-splittings are weak nonnegative of the first type, then the stationary version of the iterative method is convergent. Moreover, the unique splitting induced by matrix S is also weak nonnegative splitting of the first type.

(ii) If A is monotone and the q multi-splittings are weak nonnegative of the second type,

$E_{jl} = \alpha_{jl} I (j = 1, 2, \cdots, q,\ l = 1, 2, \cdots, p)$,

then the stationary version of the iterative method is convergent. Moreover, the unique splitting induced by matrix S is also weak nonnegative splitting of the second type, where

$$S = \prod_{j=1}^{q} \Big(\sum_{l=1}^{p} E_{q-j+1,l} (M_{q-j+1,l}^{-1} N_{q-j+1,l})^{\mu(q-j+1,l)}\Big). \quad (11)$$

Theorem 4.1 Let $A \in R^{n \times n}$ and $A^{-1} \geq 0$. Let the multisplittings

$$A = M_{1l} - N_{1l} = \cdots = M_{ql} - N_{ql}\ (l = 1, 2, \cdots, p)$$

and

$$A = P_{1l} - Q_{1l} = \cdots = P_{ql} - Q_{ql}\ \ (l = 1, 2, \cdots, p)$$

be weak nonnegative of different type and

$E_{jl} = \alpha_{jl} I (j = 1, 2, \cdots, q,\ l = 1, 2, \cdots, p)$.

If

$$M_{q-j+1,l}^{-1} \geq P_{q-j+1,l}^{-1} (j = 1, 2, \cdots, q; l = 1, 2, \cdots, p),$$

then

$$\rho(S_1) \leq \rho(S_2) < 1,$$

where

$$S_1 = \prod_{j=1}^{q} \Big(\sum_{l=1}^{p} E_{q-j+1,l} (M_{q-j+1,l}^{-1} N_{q-j+1,l})^{\mu(q-j+1,l)}\Big), \quad (12)$$

$$S_2 = \prod_{j=1}^{q} \Big(\sum_{l=1}^{p} E_{q-j+1,l} (P_{q-j+1,l}^{-1} Q_{q-j+1,l})^{\mu(q-j+1,l)}\Big). \quad (13)$$

Proof: (1) If $A = M_{1l} - N_{1l} = \cdots = M_{ql} - N_{ql}$ $(l = 1, 2, \cdots, p)$ are weak nonnegative of the first type, $A = P_{1l} - Q_{1l} = \cdots = P_{ql} - Q_{ql}$ $(l = 1, 2, \cdots, p)$ are weak nonnegative of the second type, by Lemma 4.1 there exist unique weak nonnegative splittings of the first type

$$A = B_{q-j+1,\,l} - C_{q-j+1,\,l}$$

and the second type

$$A = \tilde{B}_{q-j+1,l} - \tilde{C}_{q-j+1,l}$$

induced by

$$(M_{q-j+1,l}^{-1} N_{q-j+1,l})^{\mu(q-j+1,l)}$$

and

$$(P_{q-j+1,l}^{-1} Q_{q-j+1,l})^{\mu(q-j+1,l)},$$

respectively. It's obvious that

$$S_1 = \prod_{j=1}^{q} \sum_{l=1}^{p} E_{q-j+1,l} B_{q-j+1,l}^{-1} C_{q-j+1,l},$$

$$S_2 = \prod_{j=1}^{q} \sum_{l=1}^{p} E_{q-j+1,l} \tilde{B}_{q-j+1,l}^{-1} \tilde{C}_{q-j+1,l},$$

where

$$B_{q-j+1,l} = A(I - (M_{q-j+1,l}^{-1} N_{q-j+1,l})^{\mu(q-j+1,l)})^{-1},$$

$$\tilde{B}_{q-j+1,l} = A(I - (P_{q-j+1,l}^{-1} Q_{q-j+1,l})^{\mu(q-j+1,l)})^{-1}.$$

From

$$M_{q-j+1,l}^{-1} \geq P_{q-j+1,l}^{-1} (j = 1, 2, \cdots, q; l = 1, 2, \cdots, p)$$

and Lemma 4.4, we have

$$B_{q-j+1,l}^{-1} \geq \tilde{B}_{q-j+1,l}^{-1} (j = 1, 2, \cdots, q; l = 1, 2, \cdots, p).$$

From Lemma 4.6, we have

$$\rho(S_1) < 1,\ \rho(S_2) < 1.$$

Moreover, there exist unique weak nonnegative splittings of the first type $A = D - E$ and the second type $A = \tilde{D} - \tilde{E}$ induced by matrices S_1 and S_2, respectively. From Lemma 4.3, we have that

$$D = A(I - S_1)^{-1}, S_1 = D^{-1}E,$$
$$\tilde{D} = A(I - S_2)^{-1}, S_2 = \tilde{D}^{-1}\tilde{E}.$$

From

$$\begin{aligned}
S_1A^{-1} &= (\sum_{l=1}^{p} E_{jl}B_{jl}^{-1}C_{jl})(\sum_{l=1}^{p} E_{j-1,l}B_{j-1,l}^{-1}C_{j-1,l}) \\
&\cdots(\sum_{l=1}^{p} E_{2l}B_{2l}^{-1}C_{2l})(\sum_{l=1}^{p} E_{1l}B_{1l}^{-1}C_{1l})A^{-1} \\
&= A^{-1}(\sum_{l=1}^{p} E_{jl}C_{jl}B_{jl}^{-1})\cdots(\sum_{l=1}^{p} E_{2l}C_{2l}B_{2l}^{-1}) \\
&\quad(\sum_{l=1}^{p} E_{1l}C_{1l}B_{1l}^{-1}) \\
&= (A^{-1} - \sum_{l=1}^{p} E_{jl}B_{jl}^{-1})\cdots(\sum_{l=1}^{p} E_{2l}C_{2l}B_{2l}^{-1}) \\
&\quad(\sum_{l=1}^{p} E_{1l}C_{1l}B_{1l}^{-1}) \\
&\le (A^{-1} - \sum_{l=1}^{p} E_{jl}\tilde{B}_{jl}^{-1})\cdots(\sum_{l=1}^{p} E_{2l}C_{2l}B_{2l}^{-1})(\sum_{l=1}^{p} E_{1l}C_{1l}B_{1l}^{-1}) \\
&= (\sum_{l=1}^{p} E_{jl}\tilde{B}_{jl}^{-1}\tilde{C}_{jl})A^{-1}\cdots(\sum_{l=1}^{p} E_{2l}C_{2l}B_{2l}^{-1})(\sum_{l=1}^{p} E_{1l}C_{1l}B_{1l}^{-1}) \\
&\le (\sum_{l=1}^{p} E_{jl}\tilde{B}_{jl}^{-1}\tilde{C}_{jl})\cdots(\sum_{l=1}^{p} E_{2l}\tilde{B}_{2l}^{-1}\tilde{C}_{2l})A^{-1}\sum_{l=1}^{p} E_{1l}C_{1l}B_{1l}^{-1} \\
&\le (\sum_{l=1}^{p} E_{jl}\tilde{B}_{jl}^{-1}\tilde{C}_{jl})\cdots(\sum_{l=1}^{p} E_{2l}\tilde{B}_{2l}^{-1}\tilde{C}_{2l})(A^{-1} - \sum_{l=1}^{p} E_{1l}\tilde{B}_{1l}^{-1}) \\
&= (\sum_{l=1}^{p} E_{jl}\tilde{B}_{jl}^{-1}\tilde{C}_{jl})\cdots(\sum_{l=1}^{p} E_{2l}\tilde{B}_{2l}^{-1}\tilde{C}_{2l})A^{-1}(\sum_{l=1}^{p} E_{1l}\tilde{C}_{1l}\tilde{B}_{1l}^{-1}) \\
&= (\sum_{l=1}^{p} E_{jl}\tilde{B}_{jl}^{-1}\tilde{C}_{jl})\cdots(\sum_{l=1}^{p} E_{2l}\tilde{B}_{2l}^{-1}\tilde{C}_{2l})(\sum_{l=1}^{p} E_{1l}\tilde{B}_{1l}^{-1}\tilde{C}_{1l})A^{-1} \\
&= S_2A^{-1},
\end{aligned}$$

we have

$$\tilde{D}^{-1} - D^{-1} = (I - S_2)A^{-1} - (I - S_1)A^{-1}$$
$$= S_1A^{-1} - S_2A^{-1} \le 0.$$

By Lemma 4.5, we have that

$$\rho(S_1) \le 1,\ \rho(S_2) < 1.$$

(2) If $A = M_{1l} - N_{1l} = \cdots = M_{ql} - N_{ql} (l = 1,2,\cdots,p)$ are weak nonnegative of the second type, $A = P_{1l} - Q_{1l} = \cdots = P_{ql} - Q_{ql} (l = 1,2,\cdots,p)$ are weak nonnegative of the first type, similar to the proof of (1), we have that $\rho(S_1) \le 1$, $\rho(S_2) < 1$.

Remark 4.1 Since a nonnegative multisplitting is a weak nonnegative multisplitting of the first type and the second type, if $A^{-1} \ge 0$,

$$A = M_{1l} - N_{1l} = \cdots = M_{ql} - N_{ql}$$
$$= P_{1l} - Q_{1l} = \cdots = P_{ql} - Q_{ql} (l = 1,2,\cdots,p)$$

are all nonnegative and

$$E_{jl} = \alpha_{jl}I(j = 1,2,\cdots,q; l = 1,2,\cdots,p),$$
$$M_{q-j+1,l}^{-1} \ge P_{q-j+1,l}^{-1}(j = 1,2,\cdots,q; l = 1,2,\cdots,p),$$

then

$$\rho(S_1) \le 1,\ \rho(S_2) < 1$$

also holds.

4.2 Comparison theorems for the stationary version of Method 2

Lemma 4.7[2] Let A be a nonsingular matrix. Consider the q multisplittings defined by (6), assume that $\mu(j, k, l) = \mu(j, l) \ge 1$ for $k = 0,1,2,\cdots$.

(i) If A is monotone and the q multisplittings are weak nonnegative of the first type, then the stationary version of the iterative method is convergent. Moreover, the unique splitting induced by matrix T is also weak nonnegative splitting of the first type.

(ii) If A is monotone and the q multisplittings are weak nonnegative of the second type and $AE_l = E_lA$, then the stationary version of the iterative method is convergent. Moreover, the unique splitting induced by matrix T is also weak nonnegative splitting of the second type, where

$$T = \sum_{l=1}^{p} E_l \prod_{j=1}^{q} (M_{q-j+1,l}^{-1} N_{q-j+1,l})^{\mu(q-j+1,l)}. \quad (14)$$

Theorem 4.2 Let $A \in R^{n \times n}$ and $A^{-1} \geq 0$.. Let the multisplittings
$A=M_{1l}-N_{1l} = \cdots = M_{ql}-N_{ql}$ $(l=1,2,\cdots,p)$
and
$A=P_{1l}-Q_{1l} = \cdots = P_{ql}-Q_{ql}$ $(l=1,2,\cdots,p)$
be weak nonnegative of different type and $AE_l=E_lA$. If
$M_{q-j+1,l}^{-1} \geq P_{q-j+1,l}^{-1}(j=1,2,\cdots,q;l=1,2,\cdots,p)$,
then

$$\rho(T_1) \leq \rho(T_2) < 1.$$

where

$$T_1 = \sum_{l=1}^{p} E_l \prod_{j=1}^{q} (M_{q-j+1,l}^{-1} N_{q-j+1,l})^{\mu(q-j+1,l)}, \quad (15)$$

$$T_2 = \sum_{l=1}^{p} E_l \prod_{j=1}^{q} (P_{q-j+1,l}^{-1} Q_{q-j+1,l})^{\mu(q-j+1,l)}. \quad (16)$$

Proof: (1) If $A=M_{1l}-N_{1l} = \cdots = M_{ql}-N_{ql}$ $(l=1,2,\cdots,p)$ are weak nonnegative of the first type, $A=P_{1l}-Q_{1l} = \cdots = P_{ql}-Q_{ql}$ $(l=1,2,\cdots,p)$ are weak nonnegative of the second type, by Lemma 4.1 there exist unique weak nonnegative splittings of the first type
$A=B_{q-j+1,l}-Q_{q-j+1,l}$
and the second type
$A=\tilde{B}_{q-j+1,l}-\tilde{C}_{q-j+1,l}$
induced by
$(M_{q-j+1,l}^{-1} N_{q-j+1,l})^{\mu(q-j+1,l)}$
and
$(P_{q-j+1,l}^{-1} Q_{q-j+1,l})^{\mu(q-j+1,l)}$,
respectively. It's obvious that

$$T_1 = \sum_{l=1}^{p} E_l \prod_{j=1}^{q} B_{q-j+1,l}^{-1} C_{q-j+1,l},$$

$$T_2 = \sum_{l=1}^{p} E_l \prod_{j=1}^{q} \tilde{B}_{q-j+1,l}^{-1} \tilde{C}_{q-j+1,l},$$

where

$$B_{q-j+1,l} = A(I-(M_{q-j+1,l}^{-1} N_{q-j+1,l})^{\mu(q-j+1,l)})^{-1},$$

$$\tilde{B}_{q-j+1,l} = A(I-(P_{q-j+1,l}^{-1} Q_{q-j+1,l})^{\mu(q-j+1,l)})^{-1}.$$

From
$M_{q-j+1,l}^{-1} \geq P_{q-j+1,l}^{-1}(j=1,2,\cdots,q;l=1,2,\cdots,p)$
and Lemma 4.4, we have
$B_{q-j+1,l}^{-1} \geq \tilde{B}_{q-j+1,l}^{-1}(j=1,2,\cdots,q;l=1,2,\cdots,p)$.
From Lemma 4.7, we have
$\rho(T_1) < \rho(T_2) < 1$.

Moreover, there exist unique weak nonnegative splittings of the first type $A=D-E$ and the second type $A=\tilde{D}-\tilde{E}$ induced by matrices T_1 and T_2, respectively. From Lemma 4.3, we have that

$$D = A(I-T_1)^{-1}, T_1 = D^{-1}E,$$

$$\tilde{D} = A(I-T_2)^{-1}, T_2 = \tilde{D}^{-1}\tilde{E}.$$

From Lemma 4.2, we have that

$$
\begin{aligned}
& E_l(\prod_{j=1}^{q} B_{q-j+1,l}^{-1} C_{q-j+1,l})A^{-1} \\
&= E_l B_{q,l}^{-1} C_{q,l} B_{q-1,l}^{-1} C_{q-1,l} \cdots B_{1,l}^{-1} C_{1,l} A^{-1} \\
&= E_l B_{q,l}^{-1} C_{q,l} B_{q-1,l}^{-1} C_{q-1,l} \cdots A^{-1} C_{1,l} B_{1,l}^{-1} \\
&= E_l A^{-1} C_{q,l} B_{q,l}^{-1} C_{q-1,l} B_{q-1,l}^{-1} \cdots C_{1,l} B_{1,l}^{-1} \\
&= E_l A^{-1}(I-AB_{q,l}^{-1})(I-AB_{q-1,l}^{-1})\cdots(I-AB_{1,l}^{-1}) \\
&= E_l(A^{-1}-B_{q,l}^{-1})(I-AB_{q-1,l}^{-1})\cdots(I-AB_{1,l}^{-1}) \\
&\leq E_l(A^{-1}-\tilde{B}_{q,l}^{-1})(I-AB_{q-1,l}^{-1})\cdots(I-AB_{1,l}^{-1}) \\
&= E_l(I-\tilde{B}_{q,l}^{-1}A)A^{-1}(I-AB_{q-1,l}^{-1})\cdots(I-AB_{1,l}^{-1}) \\
&= E_l(I-\tilde{B}_{q,l}^{-1}A)(A^{-1}-B_{q-1,l}^{-1})\cdots(I-AB_{1,l}^{-1}) \\
&\leq E_l(I-\tilde{B}_{q,l}^{-1}A)(A^{-1}-\tilde{B}_{q-1,l}^{-1})\cdots(I-AB_{1,l}^{-1})\cdots \\
&\leq E_l(I-\tilde{B}_{q,l}^{-1}A)(I-\tilde{B}_{q-1,l}^{-1}A)\cdots(A^{-1}-\tilde{B}_{1,l}^{-1}) \\
&= E_l(I-\tilde{B}_{q,l}^{-1}A)(I-\tilde{B}_{q-1,l}^{-1}A)\cdots(I-\tilde{B}_{1,l}^{-1}A)A^{-1} \\
&= E_l \tilde{B}_{q,l}^{-1}\tilde{C}_{q,l}\tilde{B}_{q-1,l}^{-1}\tilde{C}_{q-1,l}\cdots\tilde{B}_{1,l}^{-1}\tilde{C}_{1,l}A^{-1} \\
&= E_l \prod_{j=1}^{q}(\tilde{B}_{q-j+1,l}^{-1}\tilde{C}_{q-j+1,l})A^{-1},
\end{aligned}
$$

so

$$
\begin{aligned}
T_1A^{-1} &= (\sum_{l=1}^{p} E_l \prod_{j=1}^{q} B_{q-j+1,l}^{-1} C_{q-j+1,l})A^{-1} \\
&= (\sum_{l=1}^{p} E_l \prod_{j=1}^{q} \tilde{B}_{q-j+1,l}^{-1} \tilde{C}_{q-j+1,l})A^{-1} = T_2A^{-1},
\end{aligned}
$$

and

$$\tilde{D}^{-1} - D^{-1} = (I - T_2)A^{-1} - (I - T_1)A^{-1}$$
$$= T_1A^{-1} - T_2A^{-1} \le 0.$$

By Lemma 4.5, we have that

$$\rho(T_1) < \rho(T_2) < 1.$$

(2) If $A=M_{1l}- N_{1l} =\cdots= M_{ql}- N_{ql}$ (l =1, 2,···, p) are weak nonnegative of the second type, $A=P_{1l}- Q_{1l} =\cdots= P_{ql}- Q_{ql}$ (l =1,2,···, p) are weak nonnegative of the first type, similar to the proof of (1), we have that

$$\rho(T_1) \le \rho(T_2) < 1.$$

Remark 4.2 Since a nonnegative multisplitting is a weak nonnegative multisplitting of the first type and the second type, if $A^{-1}\ge 0$, $A=M_{1l}- N_{1l} =\cdots= M_{ql}- N_{ql}= P_{1l}- Q_{1l} =\cdots= P_{ql}- Q_{ql}$ (l =1,2,···, p) are all nonnegative and $AE_l = E_lA$,

$$M_{q-j+1,l}^{-1} \ge P_{q-j+1,l}^{-1} (j = 1,2,\cdots,q; l = 1,2,\cdots,p),$$

then

$$\rho(T_1) \le \rho(T_2) < 1$$

also holds.

4.3 Examples

Example 4.3.1

$$A = \begin{bmatrix} 10 & -5 & -6 \\ -3 & 9 & -2 \\ -5 & 0 & 10 \end{bmatrix}$$

is a monotone matrix. Let

$$M_{11} = \begin{bmatrix} 10 & -5 & -7 \\ -3 & 12 & -2 \\ -5 & 0 & 12 \end{bmatrix}, M_{12} = \begin{bmatrix} \frac{41}{4} & -5 & -7 \\ -3 & 12 & -2 \\ -5 & 0 & 12 \end{bmatrix},$$

$$M_{21} = \begin{bmatrix} 21/2 & -5 & -7 \\ -3 & 12 & -2 \\ -5 & 0 & 12 \end{bmatrix}, M_{22} = \begin{bmatrix} \frac{43}{4} & -5 & -7 \\ -3 & 12 & -2 \\ -5 & 0 & 12 \end{bmatrix},$$

$$P_{11} = \begin{bmatrix} 11 & -5 & -6 \\ -3 & 12 & -2 \\ -6 & 0 & \frac{25}{2} \end{bmatrix}, P_{12} = \begin{bmatrix} 11 & -5 & -6 \\ -3 & 12 & -2 \\ -6 & 0 & 13 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} 11 & -5 & -6 \\ -3 & 12 & -2 \\ -6 & 0 & \frac{27}{2} \end{bmatrix}, P_{22} = \begin{bmatrix} 11 & -5 & -6 \\ -3 & 12 & -2 \\ -6 & 0 & 14 \end{bmatrix},$$

$$N_{jl} = M_{jl} - A, Q_{jl} = P_{jl} - A (j,l = 1,2),$$
$$p = q = 2, \mu(q - j + 1, l) = 1,$$

we have

$$M_{12}^{-1} = \begin{bmatrix} \frac{72}{413} & \frac{30}{413} & \frac{47}{413} \\ \frac{23}{413} & \frac{44}{413} & \frac{83}{1652} \\ \frac{30}{413} & \frac{25}{826} & \frac{54}{413} \end{bmatrix} > P_{12}^{-1} = \begin{bmatrix} \frac{52}{343} & \frac{65}{1029} & \frac{82}{1029} \\ \frac{17}{343} & \frac{107}{1029} & \frac{40}{1029} \\ \frac{24}{343} & \frac{10}{343} & \frac{39}{343} \end{bmatrix};$$

$$M_{21}^{-1} = \begin{bmatrix} \frac{72}{431} & \frac{30}{431} & \frac{47}{431} \\ \frac{23}{431} & \frac{91}{862} & \frac{21}{431} \\ \frac{30}{431} & \frac{25}{862} & \frac{111}{862} \end{bmatrix} > P_{21}^{-1} = \begin{bmatrix} \frac{108}{725} & \frac{9}{145} & \frac{164}{2175} \\ \frac{7}{145} & \frac{3}{29} & \frac{16}{435} \\ \frac{48}{725} & \frac{4}{145} & \frac{78}{725} \end{bmatrix};$$

$$M_{22}^{-1} = \begin{bmatrix} \frac{72}{449} & \frac{30}{449} & \frac{47}{449} \\ \frac{23}{449} & \frac{47}{449} & \frac{85}{1796} \\ \frac{30}{449} & \frac{25}{898} & \frac{57}{449} \end{bmatrix} > P_{22}^{-1} = \begin{bmatrix} \frac{28}{191} & \frac{35}{573} & \frac{41}{573} \\ \frac{9}{191} & \frac{59}{573} & \frac{20}{573} \\ \frac{12}{191} & \frac{5}{191} & \frac{39}{382} \end{bmatrix}.$$

and it's easy to test that

$A = M_{1l}-N_{1l}= M_{2l}-N_{2l}$ (l= 1, 2)

are weak nonnegative multisplittings of the first type, and

$A = P_{1l}-Q_{1l}= P_{2l}-Q_{2l}$ (l= 1, 2)

are weak nonnegative multisplittings of the second type.

Let

$$E_{11}=\begin{bmatrix}\frac{1}{4}&0&0\\0&\frac{1}{4}&0\\0&0&\frac{1}{4}\end{bmatrix}, E_{12}=\begin{bmatrix}\frac{3}{4}&0&0\\0&\frac{3}{4}&0\\0&0&\frac{3}{4}\end{bmatrix},$$

$$E_{21}=\begin{bmatrix}\frac{1}{5}&0&0\\0&\frac{1}{5}&0\\0&0&\frac{1}{5}\end{bmatrix}, E_{22}=\begin{bmatrix}\frac{4}{5}&0&0\\0&\frac{4}{5}&0\\0&0&\frac{4}{5}\end{bmatrix},$$

then

$$\rho(T_1)=0.1328<\rho(T_2)=0.1728<1,$$

where

$$S_1=[E_{21}(M_{21}^{-1}N_{21})+E_{22}(M_{22}^{-1}N_{22})][E_{11}(M_{11}^{-1}N_{11})+E_{12}(M_{12}^{-1}N_{12})],$$

$$S_2=[E_{21}(P_{21}^{-1}Q_{21})+E_{22}(P_{22}^{-1}Q_{22})][E_{11}(P_{11}^{-1}Q_{11})+E_{12}(P_{12}^{-1}Q_{12})].$$

Let

$$E_1=\begin{bmatrix}\frac{1}{4}&0&0\\0&\frac{1}{4}&0\\0&0&\frac{1}{4}\end{bmatrix}, E_2=\begin{bmatrix}\frac{3}{4}&0&0\\0&\frac{3}{4}&0\\0&0&\frac{3}{4}\end{bmatrix},$$

then

$$\rho(T_1)=0.1328<\rho(T_2)=0.1728<1,$$

where

$$T_1=E_1(M_{21}^{-1}N_{21})(M_{11}^{-1}N_{11})+E_2(M_{22}^{-1}N_{22})(M_{12}^{-1}N_{12}),$$

$$T_2=E_1(P_{21}^{-1}Q_{21})(P_{11}^{-1}Q_{11})+E_2(P_{22}^{-1}Q_{22})(P_{12}^{-1}Q_{12}).$$

Example 4.3.2

$$A=\begin{pmatrix}D&B&&&\\B&D&B&&\\&B&D&B&\\&&B&D&B\\&&&B&D\end{pmatrix}$$

is a monotone matrix, where

$$D=\begin{pmatrix}20&-4&&&\\-4&20&-4&&\\&\text{o}&\text{o}&\text{o}&\\&&-4&20&-4\\&&&-4&20\end{pmatrix}\in R^{n\times n},$$

$$B=\begin{pmatrix}-4&-1&&&\\-1&-4&-1&&\\&\text{o}&\text{o}&\text{o}&\\&&-1&-4&-1\\&&&-1&-4\end{pmatrix}\in R^{n\times n}.$$

$$M_{11}=\begin{pmatrix}D&E&&&\\E&D&E&&\\&E&D&E&\\&&E&D&E\\&&&E&D\end{pmatrix},$$

$$E=\begin{pmatrix}-3.5&-1&&&\\-1&-3.5&-1&&\\&\text{o}&\text{o}&\text{o}&\\&&-1&-3.5&-1\\&&&-1&-3.5\end{pmatrix}\in R^{n\times n},$$

$$P_{11}=\begin{pmatrix}D&F&&&\\F&D&F&&\\&F&D&F&\\&&F&D&F\\&&&F&D\end{pmatrix},$$

$$F=\begin{pmatrix} -3 & -1 & & & \\ -1 & -3 & -1 & & \\ & \text{o} & \text{o} & \text{o} & \\ & & -1 & -3 & -1 \\ & & & -1 & -3 \end{pmatrix}\in R^{n\times n},\qquad J=\begin{pmatrix} -1.5 & -1 & & & \\ -1 & -1.5 & -1 & & \\ & \text{o} & \text{o} & \text{o} & \\ & & -1 & -1.5 & -1 \\ & & & -1 & -1.5 \end{pmatrix}\in R^{n\times n},$$

$$M_{12}=\begin{pmatrix} D & G & & & \\ G & D & G & & \\ & G & D & G & \\ & & G & D & G \\ & & & G & D \end{pmatrix},\qquad P_{21}=\begin{pmatrix} D & K & & & \\ K & D & K & & \\ & K & D & K & \\ & & K & D & K \\ & & & K & D \end{pmatrix},$$

$$G=\begin{pmatrix} -2.5 & -1 & & & \\ -1 & -2.5 & -1 & & \\ & \text{o} & \text{o} & \text{o} & \\ & & -1 & -2.5 & -1 \\ & & & -1 & -2.5 \end{pmatrix}\in R^{n\times n},\qquad K=\begin{pmatrix} -1 & -1 & & & \\ -1 & -1 & -1 & & \\ & \text{o} & \text{o} & \text{o} & \\ & & -1 & -1 & -1 \\ & & & -1 & -1 \end{pmatrix}\in R^{n\times n},$$

$$P_{12}=\begin{pmatrix} D & H & & & \\ H & D & H & & \\ & H & D & H & \\ & & H & D & H \\ & & & H & D \end{pmatrix},\qquad M_{22}=\begin{pmatrix} D & L & & & \\ L & D & L & & \\ & L & D & L & \\ & & L & D & L \\ & & & L & D \end{pmatrix},$$

$$H=\begin{pmatrix} -2 & -1 & & & \\ -1 & -2 & -1 & & \\ & \text{o} & \text{o} & \text{o} & \\ & & -1 & -2 & -1 \\ & & & -1 & -2 \end{pmatrix}\in R^{n\times n},\qquad L=\begin{pmatrix} -0.5 & -1 & & & \\ -1 & -0.5 & -1 & & \\ & \text{o} & \text{o} & \text{o} & \\ & & -1 & -0.5 & -1 \\ & & & -1 & -0.5 \end{pmatrix}\in R^{n\times n},$$

$$M_{21}=\begin{pmatrix} D & J & & & \\ J & D & J & & \\ & J & D & J & \\ & & J & D & J \\ & & & J & D \end{pmatrix},\qquad P_{22}=\begin{pmatrix} D & X & & & \\ X & D & X & & \\ & X & D & X & \\ & & X & D & X \\ & & & X & D \end{pmatrix},$$

and it's easy to test that
$A = M_{1l} - N_{1l} = M_{2l} - N_{2l} (l = 1, 2)$
are weak nonnegative multisplittings of the first type, and
$A = P_{1l} - Q_{1l} = P_{2l} - Q_{2l} (l = 1, 2)$
are weak nonnegative multisplittings of the second type.
Let

$$E_{11} = \frac{1}{4} I, E_{12} = \frac{3}{4} I,$$
$$E_{21} = \frac{1}{5} I, E_{22} = \frac{4}{5} I,$$

then

$$\rho(S_1) = 0.4277 < \rho(S_2) = 0.5139 < 1,$$

where

$$S_1 = [E_{21}(M_{21}^{-1} N_{21}) + E_{22}(M_{22}^{-1} N_{22})]$$
$$[E_{11}(M_{11}^{-1} N_{11}) + E_{12}(M_{12}^{-1} N_{12})],$$
$$S_2 = [E_{21}(P_{21}^{-1} Q_{21}) + E_{22}(P_{22}^{-1} Q_{22})]$$
$$[E_{11}(P_{11}^{-1} Q_{11}) + E_{12}(P_{12}^{-1} Q_{12})].$$

Let

$$E_1 = \frac{1}{4} I, E_2 = \frac{3}{4} I,$$

then

$$\rho(T_1) = 0.4291 < \rho(T_2) = 0.5138 < 1,$$

where

$$T_1 = E_1(M_{21}^{-1} N_{21})(M_{11}^{-1} N_{11})$$
$$+ E_2(M_{22}^{-1} N_{22})(M_{12}^{-1} N_{12}),$$
$$T_2 = E_1(P_{21}^{-1} Q_{21})(P_{11}^{-1} Q_{11})$$
$$+ E_2(P_{22}^{-1} Q_{22})(P_{12}^{-1} Q_{12}).$$

5. CONCLUDING REMARKS

In this paper, we give some new results on parallel synchronous alternating iterative methods for solving the system of linear equations $Ax = b$. In section 3, we present two convergence theorems for the parallel synchronous alternating methods when the coefficient matrix is a nonsingular H-matrix. In section 4, we present some comparison theorems of the spectral radius of the iterative matrix when the coefficient matrix is a monotone matrix, and we present two numerical examples to confirm our theoretical results.

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