Multifractality of Daily Rainfall in Thailand

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ABSTRACT: It has long been known that temporal rainfall intensities, among other geophysical data, are very irregular. Since the introduction of multifractal property, the presence of multifractality in rainfall data with various scale ranges has been confirmed in many areas with different climates. Using the Wavelet Transform Modulus Maxima (WTMM) method, we compute the spectra of singularities of daily rainfall from April 1992 to March 2002 collected at 14 stations in the eastern, central and southern parts of Thailand. Based on the supports of their spectra, we can then loosely categorize the stations into 4 groups: A) Most irregular station: Chumphon with spectrum support in [-0.8,+1.0]; B) Most regular stations: Prachinburi with spectrum support in [-0.28,+2.11], Ranong and Chanthaburi; C) Typical stations: these are stations with spectrum supports in [-0.5,+1.5] which consists of Donmuang, Lopburi, Kanchanaburi, Phetchaburi, Chonburi, Nakhonsithammarat and Rayong; and D) Stations with very incomplete spectrum: the decreasing parts of the spectra of these stations are missing. They are Narathiwat, Phuket and Prachuapkhirikhan. Further study further is required to find the geographical and/or meteorological reasons behind this result.

Key words: Thailand, rainfall, multifractal, multifractality, spectrum of singularities, wavelet transform modulus maxima (WTMM)

INTRODUCTION

Temporal rainfall is one of the real-world observed data whose pointwise regularities change wildly from point to point. These strongly irregular fluctuations in the pattern of rainfall make it difficult and almost impossible to study the temporal structure of rainfall by classical models. However, the advent of multifractals and multiplicative cascades(17) have opened up a new path to the study of rainfall.(9) Multifractal functions are, by definition, functions whose points with given regularities are so intertwined that for any regularity exponent in a nonempty open interval, there are sufficiently many points with the regularity that they form a set of positive dimension. The mapping of the regularity exponent to this corresponding dimension is called the spectrum of singularities.

In this article, the multifractality of temporal rainfall data from across many stations in Thailand is studied. As the presence of the multifractal scaling in temporal rainfall has already been established at a reasonably diverse array of locations in the literature, e.g. the (daily) rainfall at Vale Formoso in Portugal,(13) at Valentia in Ireland,(12) and in Rio de Janeiro state in Brazil,(14) here we omit the multifractal scaling study and only compute the spectra of singularities of the daily rainfall.

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MATHEMATICAL BACKGROUNDS

The *pointwise Hölder exponent* $h_f(x_0)$ of a function $f : R \to R$ at a point x_0 is the supremum of non-integer $\alpha \in R$ for which $f \in C^\alpha(x_0)$, which means that there exist a polynomial P of degree less than α and a constant C such that

$$\left|f(x) - P(x - x_0)\right| \le C\left|x - x_0\right|^\alpha \quad \dots (1)$$

for all x in a neighborhood of x_0. The exponent $h_f(x_0)$ reflects the degree of regularity of f at x_0. In mathematics, one usually encounters functions which attain only a few pointwise Hölder exponents over their domain. On the contrary, the pointwise regularities of real-world data such as stock prices, temperatures and rainfall range over a full interval and often change wildly from point to point.

For each α, the set of all points whose pointwise exponent of such real-world data is equal to α are often so intertwined that a suitable characteristic of such sets is the Hausdorff dimension.(16) Hence we define the *spectrum of singularities* of f as the function d_f mapping an exponent α to the Hausdorff dimension of the set of all points having the same pointwise Hölder exponent α, i.e. $d_f(\alpha) = \dim\{x : h_f(x) = \alpha\}$. By convention, the dimension of the empty set is $-\infty$.

A *multifractal function* is defined to be a function whose spectrum of singularities is supported in a set containing a nonempty open interval. Here, the *support* of d is the closure of the set $\{\alpha : d(\alpha) > 0\}$. Multifractal functions serve as our mathematical representations of signals/data whose regularities change wildly from point to point.

Note that the definition of the spectrum of singularities involves a series of intricate mathematical operations starting from computing the pointwise exponent at each point to computing the Hausdorff dimension whose definition itself is not robust to evaluate numerically. See.(16) There have been many procedures proposed over the past fifteen years to circumvent these difficulties.(2,3,8) One successful approach(2,3) takes advantage of the relationship between the pointwise Hölder exponent and the decay rate of its continuous wavelet transform for small scales. Let us recall the definition of wavelet functions used in this study and their properties. See(8) for more details.

Let $\psi \in L^2(R)$ be a *wavelet function* satisfying the admissibility condition,

$$\int_0^\infty \left|\hat{\psi}(\xi)\right| \frac{d\xi}{\xi} = 1 \text{ and } \hat{\psi}(\xi) = 0 \text{ for all } \xi < 0.$$

We also assume that ψ has N vanishing moments, i.e.

$$\int x^n \psi(x)\,dx = 0 \quad \text{for all } 0 \le n < N.$$

Suppose further that ψ is well-localized in both space and frequency, that is for any $k = 0, \dots, N$ and $m \in \mathbb{N}$ there is a constant C_m ($C_{m,k}$) for which;

$$\left|\psi^{(k)}(t)\right| \le \frac{C_m}{1 + |t|^m} \quad \text{for all } t \in R.$$

An example of such a wavelet function is the N^{th} derivative, denoted by g_N, of the Gaussian function $g_0(x) = e^{-x^2/2}$. The continuous wavelet transform of $f \in L^\infty(R)$ with respect to the wavelet ψ is defined by:

$$W_\psi f(b,a) = \frac{1}{a}\int_{\mathbb{R}} f(x)\overline{\psi}\left(\frac{x-b}{a}\right)dx$$

for $b \in R$ and $a > 0$.

In 1991, Jaffard(1) discovered that the pointwise Hölder exponents of a function can be characterized by some decay properties of its wavelet transform with respect to a well-chosen wavelet.

Theorem 1. Let $f \in L^\infty(R)$ and ψ be a wavelet function with N vanishing moments and its derivatives up to order N have fast decay.

If f is pointwise Hölder regular with exponent $\alpha \le N$ at a point x_0, then there exists a constant C such that, for all $b \in R$ and $a > 0$;

$$\left|W_\psi f(b,a)\right| \le C a^{\alpha + \frac{1}{2}}\left(1 + \frac{\left|b - x_0\right|^\alpha}{a^\alpha}\right).$$

Conversely, if $\alpha < N$ is not an integer and there is a constant C and $\beta < \alpha$ such that:

$$\left|W_\psi f(b,a)\right| \le C a^{\alpha + \frac{1}{2}}\left(1 + \frac{\left|b - x_0\right|^\beta}{a^\beta}\right)$$

for all $b \in R$ and $a > 0$, then f is a pointwise Hölder regular with exponent α at x_0.

In 1992, Mallat and Hwang[7] introduced the *wavelet transform modulus maxima* (WTMM) method, to estimate pointwise Hölder exponents via the decay rate of the wavelet transform along maxima lines. Maxima lines are connected curves consisting of, at each scale a, local maxima of $|W_\psi f(x,a)|$ considered as a function of x. In order to compute its spectrum of singularities, Muzy et al.[2,3] then defined the partition function for $a > 0$ and $q \in R$ by:

$$Z(q,a) = \sum_{\substack{l \in L(a) \\ (x,a) \in l}} |W_\psi f(x,a)|^q$$

where each $l \in L(a)$ is a maxima line consisting of (x,a') for $0 < a' \le a$. As a decreases to 0, the exponent $\tau(q)$ is defined from the power law behavior of the partition function via, $Z(q,a) \sim a^{\tau(q)}$, which means that:

$$\tau(q) = \liminf_{a \to 0^+} \frac{\log Z(q,a)}{\log a}, \quad q \in R.$$

The spectrum of singularities D_f of f can then be computed via the Legendre transform of $\tau(q)$:

$$D_f(h) = \min_{q \in \mathbb{R}} (qh - \tau(q)). \qquad \ldots (2)$$

This formula has been used to compute the spectrum of singularities of many empirical data, such as stock prices, heart beats and rainfall. [4,5,9,10,18,19]

RAINFALL DATA

Daily rainfall data from April 1992 to March 2002 (which shall be referred to as the rain years 1992-2001 and consists of 3652 days) at 14 stations in the central, eastern and southern parts of Thailand were obtained from the Thai Meteorology Department (TMD) and are used in our study. Their station names, coordinates, elevations and basic statistics are shown in Table 1. Our choice of stations and time period is a result of comparing the monthly data from TMD against the monthly data from the Royal Irrigation Department (RID) as daily rainfall data from the RID is not available to us. We actually first set out to analyze the spectra of singularities of daily rainfall at 19 stations with elevation not exceeding 100 meters. But monthly data comparison forces us to eliminate Phitsanulok, Nakhonsawan, Songkhla and Krabi from our study as there are significant differences scattering throughout the whole periods, especially the chosen period of the rain years 1992-2001. We also have to discard Pattani as the data for the whole month of November 1998 are missing. This leaves us with 14 stations where most differences cluster in the first several decades since the rainfall data were first collected in 1951-1952. Hence, we have to choose the aforementioned period of ten years in this study. The last column in Table 1 shows the square root of the sum square of the relative errors of monthly data from TMD and RID during the rain years 1992-2001. The maximum "total relative errors" is 0.33 which amounts to only <0.3% in relative error each month. Only two stations, Chanthaburi and Narathiwat, have more than 10 months where the monthly rainfalls disagree. It should be noted that any trace of rain below 0.1 mm is considered 0 mm.

For the last three columns of Table 1 and all numbers in Table 2, the values which are among the highest five percent are in boldface, while the lowest five percent are underlined.

The annual rainfall, monthly rainfall and average monthly rainfall, observed for the rain years 1992-2001, at two typical stations are shown in Figure 1.

DATA ANALYSIS AND RESULTS

We analyze the multifractality of daily rainfall from April 1992 to March 2002 at the 14 stations listed in Table 1. Their spectra of singularities are computed according the WTMM method by the software package LastWave.[15]

LastWave is a wavelet-oriented signal processing software written in C by Emmanuel Bacry and coauthors out of their dissatisfaction of then-current signal processing softwares. It consists of a powerful command line language which includes matlab-like numerical facilities with high level structures (such as signals, images and wavelet transforms) and a high level object-oriented graphic language which allows the display of both some simple structures (e.g., buttons, strings, text using any font, etc) and some complex structures (signals, images, wavelet transforms, extreme representation, etc). In this project, we used LastWave version 2.0.1 running on Windows XP with CygWin. See [5,9,10,11] for more details.

Table 1. Stations, their coordinates, elevations and basic statistics for the time interval 1992/4/1 - 2002/3/31 (120 months).

	Station	Latitude	Longitude	Elev (m)	Zero rainfall days (%)	Mean (mm)/ coefficient of variation of daily rainfall	Total rel. error /# errors (mo's)*
1	Kanchanaburi (C)	14° 1′ N	99 32′ E	28	**70.35**	2.82/3.29	0.04/3
2	Lopburi (C)	14° 48′ N	100° 37′ E	10	**72.62**	2.96/3.16	0.01/4
3	Donmuang (C)	13° 55′ N	100° 36′ E	4	68.67	3.87/3.01	0.05/3
4	Prachinburi (E)	14° 3′ N	101° 22′ E	4	64.43	4.67/2.71	0.21/10
5	Chanthaburi (E)	12° 37′ N	102° 7′ E	3	54.57	7.95/2.28	0.07/**12**
6	Rayong (E)	12° 38′ N	101° 21′ E	3	67.14	4.13/2.94	**0.33**/9
7	Chonburi (E)	13° 22′ N	100° 59′ E	1	68.10	3.46/2.91	0.07/4
8	Phetchaburi (SE)	13° 9′ N	100° 14′ E	2	**72.70**	2.70/**3.85**	0.01/2
9	Prachuapkhirikhan (SE)	11° 50′ N	99° 50′ E	4	67.85	2.99/3.49	0.10/7
10	Chumphon (SE)	10° 29′ N	99° 11′ E	3	55.37	5.37/2.60	0.01/8
11	Nakhonsithammarat (SE)	8° 28′ N	99° 58′ E	7	52.46	7.12/2.80	0.04/7
12	Narathiwat (SE)	6° 25′ N	100° 49′ E	2	53.34	7.77/2.74	0.11/**13**
13	Phuket (SW)	7° 58′ N	98° 24′ E	2	51.51	5.96/2.27	0.17/9
14	Ranong (SW)	9° 59′ N	98° 37′ E	6	46.06	**11.68**/2.00	0.09/8

Figure 1. Annual, monthly and average monthly rainfall at Rayong and Chumphon.

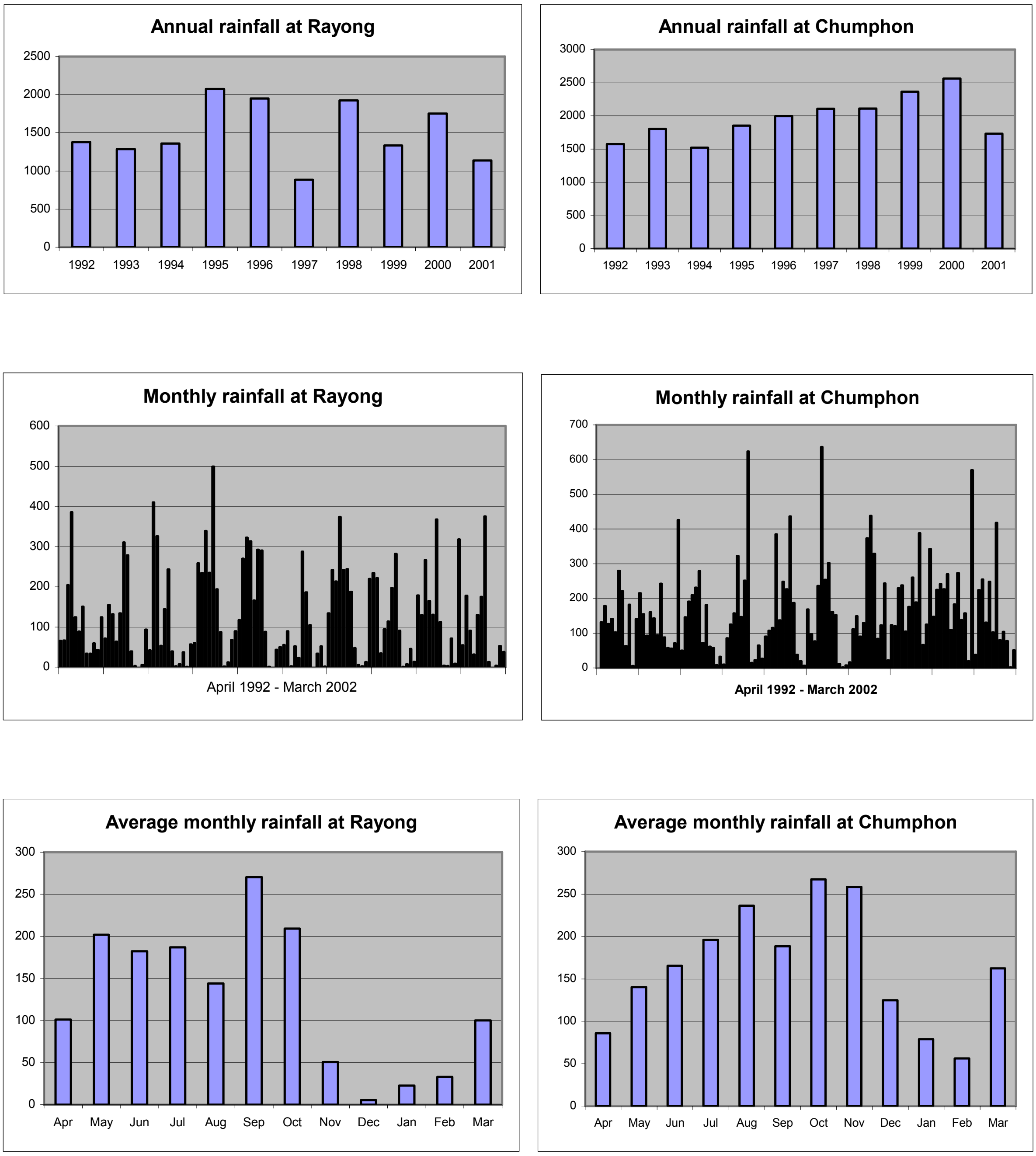


Table 2. Key exponents and maximum dimensions of the spectra of singularities.

Station	h_{min}	h_{center}	h_{max}	$h_{max} - h_{min}$	$D(h_{center})$
Kanchanaburi (C)	-0.65	+0.11	< 1.7	< 2.4	0.96
Lopburi (C)	~ -0.5	+0.10	1.60	2.1	0.99
Donmuang (C)	-0.44	+0.17	1.65	2.1	1.00
Prachinburi (E)	**-0.28**	**+0.24**	**2.11**	2.4	1.02
Chanthaburi (E)	-0.43	**+0.23**	1.94	2.4	**1.07**
Rayong (E)	-0.61	+0.04	< 1.5	< 2.1	1.00
Chonburi (E)	-0.57	+0.11	< 1.6	< 2.2	1.00
Phetchaburi (S)	-0.66	+0.08	< 1.4	< 2.1	0.96
Prachuapkhirikhan (S)	< -0.60	-0.03	--	--	0.97
Chumphon (S)	-0.80	-0.20	< 1	< 1.8	1.01
Nakhonsithammarat (S)	-0.64	+0.06	< 1.4	< 2.1	1.03
Narathiwat (S)	-0.53	+0.05	--	--	1.04
Phuket (S)	-0.59	-0.03	--	--	1.04
Ranong (S)	-0.47	**+0.24**	**2.14**	**2.6**	**1.07**

In the computation of the continuous wavelet transform (CWT), we have chosen to use the third derivative of the Gaussian function (g3) and the following scale parameters: minimum scale (aMin) = 1, number of octaves (nOct) = 5, number of voices per octave (nVoice) = 10. The two boundaries of the data are padded by zeros. For the partition function $Z(q,a)$, we selected the list of the moments q for each station so that the absolute value of the Pearson correlation coefficient of $\log a$ and $Z(q,a)$ is at least 0.9 and that the resulting spectrum never takes negative values. Before settling for g3 wavelet, we have compared the spectra computed using g2, g3 and g4, i.e. second, third and fourth derivatives of the Gaussian function and found that for a typical station they are quite close in the increasing part (corresponding to positive q 's) but start to diverge in the decreasing part (corresponding to negative q 's). There are, of course some exceptions. See Figure 2.

The maximum dimensions $D(h_{center})$ for all the 14 stations lay between 0.96 and 1.07. Theoretically, since the underlying dimension is one, the spectra of singularities D have to be bounded above by 1. These overshoots are caused by numerical errors and/or the numerical algorithm (WTMM) chosen to compute the spectra. We observed that Ranong, Chanthaburi and Phuket whose maximum dimensions are 1.07, 1.07 and 1.04, respectively, have the largest gaps among their spectra, in terms of both the increasing and decreasing parts, analyzed by the 2nd (g2), 3rd (g3) and 4th (g4) derivatives of the Gaussian. See Figure 2. So the computed spectra at these three stations should be considered less reliable and used with care. Narathiwat, with a maximum dimension of 1.04, also exhibits large gaps on the decreasing part between spectra analyzed by the three wavelets. Surprisingly, the increasing parts are very close up to the point where $h = 0$. Narathiwat and Phuket are the two stations with the shortest spectra on the decreasing part (negative q 's) due to low correlation coefficients of $\log a$ and $\log Z(q,a)$. See Figure 3.

In Table 2, h_{min} = the minimum exponent, h_{mix} = the maximum exponent and h_{center} = the exponent with maximum dimension.

From Figure 3, it is evident that the rainfall at Chumphon is by far the most singular as both the minimum regularity (-0.8) and the center of singularities (-0.2) are the lowest. Moreover, the support of its spectrum is one of the narrowest. Therefore, in our study, the daily rainfall at Chumphon possesses the highest variability. For the other extreme, Prachinburi possesses the weakest singularities in the interval [-0.28, 2.11]. Ranong, Chanthaburi, Donmuang and Lopburi form a group of four runners-up with least regularity exponents between -0.5 and -0.4. Furthermore, their highest regularity exponents are 2.14, 1.94, 1.65 and 1.60, respectively. These are exactly the five stations that do not display a problem with low correlation coefficients when fitting $\log a$ versus $\log Z(q,a)$ for negative q's.

Ranong, Prachinburi and Chanthaburi have the widest singularity support making them the most multifractal and hence the most unpredictable. They also possess the highest h_{center} values at 0.24, 0.24 and 0.23, respectively. Despite their different geographical locations, the rainfall variability at Ranong, Prachinburi and Chanthaburi are quite similar. It would then be very interesting to find out whether these areas have any common physical characteristics.

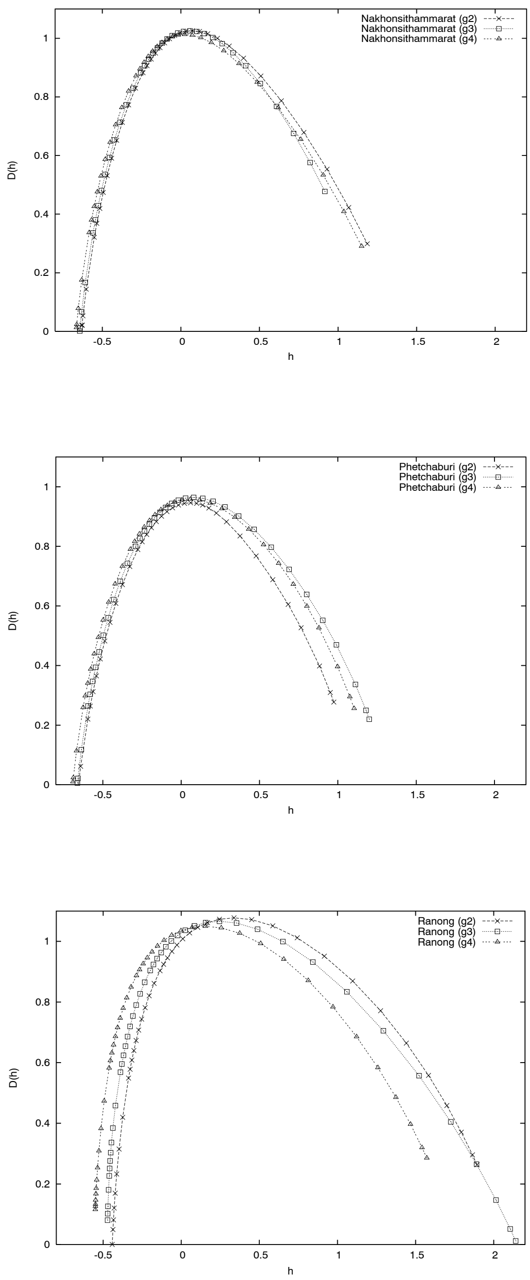
CONCLUDING REMARKS

Through comparison of the spectra of singularities of daily rainfall from April 1992 to March 2002, Chumphon was found to be the most irregular whilst Prachinburi was the most regular. This means that, of all the fourteen stations in this study, rainfall at Chumphon station tends to vary more sharply from day to day but Prachinburi usually has a smoother variation in the daily rainfall. Even though the close seconds to Prachinburi are Ranong and Chanthaburi, one should not make such a conclusion due to the low reliability of their spectra. Low reliability aside, rainfall variability at Prachinburi, Ranong and Chanthaburi are very similar and the most multifractal.

Based on the support of their spectra of singularities, we can loosely categorize the 14 stations into four groups:

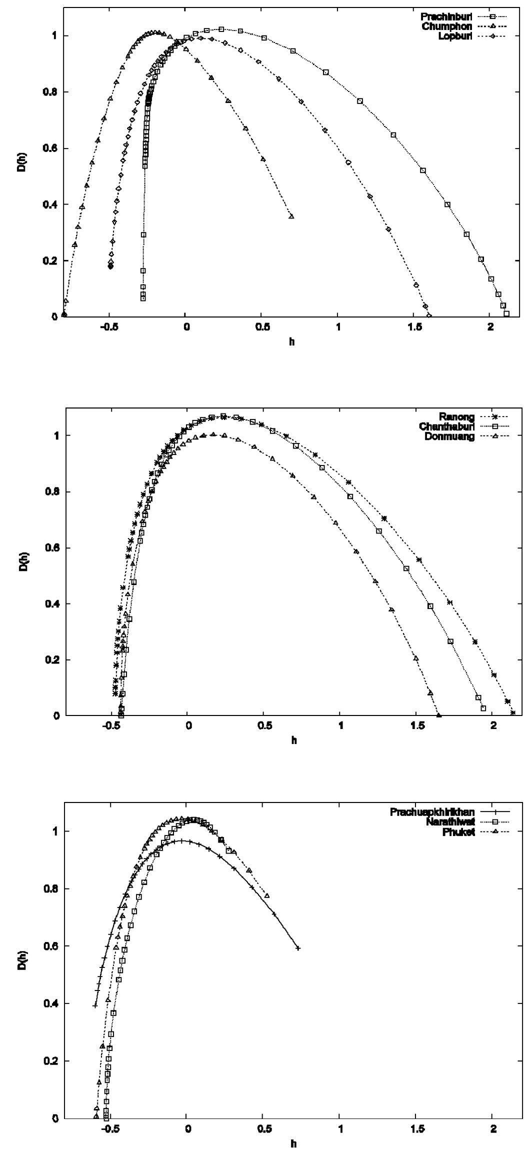
A) **Most irregular**: Chumphon
B) **Most regular**: Prachinburi, Ranong and Chanthaburi
C) **Typical**: Donmuang, Lopburi, Kanchanaburi, Phetchaburi, Chonburi, Nakhonsithammarat and Rayong
D) **Very incomplete spectrum**: Narathiwat, Phuket and Prachuapkhirikhan

Figure 2. The spectra of singularities using the three wavelets (g2, g3, g4) for data from Nakhonsithammarat, Phethchaburi and Ranong: These three stations are representatives of the best, a typical and the worst, in terms of the closeness of the spectra.



The spectrum at Prachinburi has a very unique shape as the increasing part is extremely steep and almost vertical. It would be worthwhile to study and determine which characteristics of rainfall contribute to such behavior.

Figure 3. The spectra of singularities (using g3) at nine different locations.



From Table 1 and 2, there is no clear relationship between basic statistics and spectra except, possibly, for Ranong and Chanthaburi. They have the highest daily rainfall averages, very low coefficients of variation and at the same time are two of the most multifractal. Note that these relationships are obviously not true for general data.

For comparison purposes, spectra of every-other-day rainfall or weekly rainfall should also be computed. But our short data length of just 3652 does not allow us to do so.

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