Mathematical Analysis of Stochastic Regularization Approach for Super-Resolution Reconstruction

Vorapoj Patanavijit

Department of Computer and Network Engineering, Faculty of Engineering
Assumption University, Bangkok, Thailand
E-mail: <Patanavijit@yahoo.com>

Abstract

Traditionally, several distorting processes affect the quality of image sequences or video acquired by commercial digital cameras. Some of the more important distorting effects include warping, blurring, down sampling and additive noise. The term SRR (Super-Resolution Reconstruction) ranges from blur removal by deconvolution in single image to the creation of a single high resolution image from multiple low resolution images having relative sub-pixel displacements. In all cases, the goal of SRR is to remove the effect of possible blurring and noise in the LR images and to obtain images with resolutions that go beyond the conventional limits of the uncompensated imaging system. Thus, the major advantage of this approach is that the cost of implementation is reduced and the existing low resolution (LR) imaging systems can still be utilized. Due to the importance of SRR research and the advantages of the SRR algorithm, this article aims to review the mathematical analysis of the SRR algorithm based on stochastic regularization approaches, one of the most popular techniques introduced by the SRR research community during the last two decades. The mathematical models of SRR algorithm based on classical L1 and L2 norms with several classical regularization functions are comprehensively derived. Finally, the mathematical solutions of each case are obtained by the classical systematical approach.

Keywords: *Digital image processing, digital image reconstruction, stochastic regularization, Laplacian regularization.*

Introduction

Due to several advantages of Super-Resolution Reconstruction (SRR), applications for the techniques of SRR from image sequences grow rapidly as the theory gains exposure. Continuing researches and the availability of fast computational machineries have made these methods increasingly attractive in applications requiring the highest restoration performance. SRR techniques have already been applied to problems in a number of applications such as satellite imaging, astronomical imaging, video enhancement and restoration, video standards conversion, confocal microscopy, digital mosaicing, aperture displacement cameras, medical imaging, diffraction tomography and video freeze frame (Kang and Chaudhuri 2003; Ng and Bose 2003; Park *et al*. 2003; Rajan *et al*. 2003; Patanavijit 2009).

Usually, the SRR is a process that attempts to reconstruct or recover an image that has been degraded by using some *a priori* knowledge of the degradation phenomenon. Therefore, SRR algorithms are oriented toward modeling the degradation and applying the inverse process in order to recover the original image (Gonzalez and Woods 1992).

In the next section, a succinct overview of some basic concepts in mathematical inverse problems is provided. The discussion there is nontechnical so that the reader may obtain an intuitive comprehension of the fundamental ideas prior to the presentation of greater detail in subsequent sections.

Inverse Problems

Definition of an Inverse Problem: The inverse problem can be defined as follows (Borman 2004):

"We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other has never been studied and is not so well understood. In such cases, the former is called the direct problem, while the latter is the inverse problem."

This description hints at the arbitrariness of the definition as to which problem is considered direct and which is the inverse. Consider the following classical example to clarify the terminology. This well known example of a direct/inverse problem pair concerns polynomials. Given a polynomial of order *p*, the direct problem is finding the *p* roots of the polynomial. The corresponding inverse problem is finding the polynomial given the *p* roots. In the case of polynomials, it is clear that the direct problem is more difficult.

The above example illustrates another typical characteristic of direct/inverse problem pairs. The data for the direct problem is the desired solution of the inverse problem and vice versa.

Well-Posed and Ill-Posed Problems: Hadamard (cited by Borman 2004), in his work on differential equations, classified a problem as "well-posed" if the solution to the problem has the following characteristics:

1. Solution Existence: The solution of the problem or model must exist. There may be no model that exactly fits the data and therefore no solution does exits for the approximated model. This can occur in practice because the mathematical model of the system's physics is approximated or because the data contain noise.

2. Solution Uniqueness: If the solution of the problem or model exists, then the solution must be unique. For some models, though exact solutions do exits, they may not be unique, even for an infinite number of exact data points. Non-unique solution is a characteristic of rank-deficient discrete linear inverse problems.

3. Stability of the Solution Process (Continuous Dependence of the Data): The solution of the problem must depend on the data. The process of computing an inverse solution can be, and often is, extremely unstable in that a small change in measured/observed data can lead to an enormous change in the estimated model.

In contrast, a problem which fails to satisfy any of the Hadamard conditions is said to be "ill-posed". SRR is to be considered ill-posed due to at least one of the following reasons: (i) Solution nonexistence; (ii) Solution non-uniqueness; (iii) Instability of the solution process.

In fact, there is always loss of information due to the observation process. Therefore, the information content of the solution is lower than that of the original information. This irrecoverable loss of information does not present significant difficulties for the direct problem. However, if the objective is to determine the original information from the observed information, that is, to solve the inverse problem, this loss of information has serious consequences. It is not possible to reverse the process exactly and return to the original information due to the loss of information. The inverse problem fails to obtain a unique solution.

Regularization Techniques and the Solution to Ill-Posed Problems: Regularization is a term which refers to methods which utilize additional information to compensate for the loss of information which characterizes ill-posed problems. This additional information is typically referred to as *a priori* or prior information as it cannot be derived from the observations or the observation process and must be known in advance "before the fact". Typically, the a priori information is chosen to represent desired characteristics of the solution, for example, total energy, smoothness, positivity and so on. The role of the *a priori* information is to constrain or reduce the space of solutions which are compatible with the observed data.

A deterministic theory of regularized solutions to ill-posed problems was pioneered by A.N. Tikhonov (cited by Borman 2004).

Inverse Problems of Super-Resolution Reconstruction

Super-Resolution Reconstruction as an Ill-Posed Inverse Problem: In SRR, typically, the low-resolution (LR) images represent different "looks" at the same scene (Park *et al*. 2003). That is, LR images are subsampled (aliased) as well as shifted with sub-pixel precision. If the LR images are shifted by integer units, then each image contains the same information, and thus there is no new information that can be used to reconstruct a high-resolution (HR) image. If the LR images have different sub-pixel shifts from each other and aliasing is present, however, then each image cannot be obtained from others. In this case, the new information contained in each LR image can be exploited to obtain a HR image. To obtain different looks at the same scene, some relative scene motions must exist from frame to frame via multiple scenes or video sequences. Multiple scenes can be obtained from one camera with several captures or from multiple cameras located in different positions. These scene motions can occur due to the controlled motions in imaging systems, e.g., images acquired from orbiting satellites. The same is true for uncontrolled motions, e.g., movement of local objects or vibrating imaging systems. If these scene motions are known or can be estimated within sub-pixel accuracy and one combines these LR images, then SRR is possible. One of the recurring issues in this work is that multiframe SRR is usually an ill-posed inverse problem. (Borman 2004)

Super-Resolution Reconstruction is an Inverse Problem: SRR refers to the restoration of a sequence of observed LR images that has information content beyond the spatial and/or temporal band limit of the imaging system (bandwidth extrapolation). Hence, the corresponding inverse problem is that of determining estimate(s) of the scene given the observed image sequence and the characterization of the imaging process. Given the characteristics of the imaging process and system, the forward problem is the simulation, while the inverse problem is the restoration (Borman 2004).

Super-Resolution Reconstruction is an Ill-Posed Problem: Recall that ill-posedness implies failure of one or more of the Hadamard conditions. The multiframe SRR problem may fail to satisfy one or more of these conditions. The failure may result from either the characteristics of the imaging system or the observed data (Borman 2004).

1. Nonexistence of the solution: the presence of noise in the observation process may result in an observed image sequence which, given the imaging system characterization, is inconsistent with any scene. The result is that the system is noninvertible and the scene cannot be estimated from the observations.

2. Non-uniqueness of the solution: when the operator which characterizes the imaging process is many-to-one, there exists a nontrivial space of solutions consistent with any given observed image sequence, that is, the solution to the inverse problem is non-unique. For example, in band-limited imaging systems, all out-of-band scene data represent the null space of the imaging process operator. Even if the imaging operator is nonsingular, a simple lack of data, which represent constraints on the solution space, is sufficient to result in the non-uniqueness of the solution. For example, consider a discretized imaging scenario with P observed LR images each consisting of N pixels. These observed data provide a maximum of PN independent constraints. Assume that a single super-resolution image containing $M > PN$ pixels is to be estimated from the data. Since the number of unknowns exceeds the number of constraints, it is clear that there are insufficient constraints for the existence of a unique solution to the inverse problem. Furthermore, since super-resolution, by definition, requires the restoration of information that is lost in the imaging process, it should be expected that the solution to the super-resolution restoration problem is likely to be non-unique.

3. Discontinuous dependence of the solution on the data: depending on the characteristics of the imaging system, the inverse problem may be highly sensitive to perturbations of the data. For example, consider an imaging system with a spectral response which decreases asymptotically toward zero with increasing frequency. While such a system is invertible in theory, in practice the inverse is unstable. An arbitrarily small noise component at a sufficiently high frequency leads to an arbitrarily large spurious signal in the computed restoration. In practice, such restorations are typically overwhelmed by the amplification of the noise.

While, in rare circumstances, it happens that the Hadamard conditions are satisfied, in general, practical applications involving multiframe SRR are invariably ill-posed. Despite the difficulties caused by the ill-posedness, regularized solution methods enable high quality SRR as is shown in later sections. The inclusion of *a priori* information is crucial to achieving this.

Stochastic Regularization Approach for SRR Algorithm

In this section, the classical SRR algorithm is presented. First, the SRR observation model is described and, consequently, the classical regularized ML for the SRR algorithm is stated.

The SRR Observation Model

The first step to comprehensively analyze the SRR problem is to formulate an observation model that relates the original HR image to the observed LR images. The observation models can be broadly divided into the models for still images and for video sequence. To present a basic concept of SRR algorithms, mathematical analysis is first employed in the observation model and, later, the mathematical model is solved with different regularizations. (Elad and Feuer 1997; Elad and Feuer 1999b)

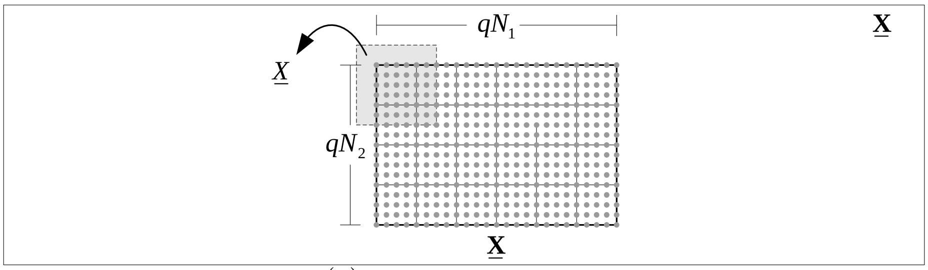
Define a low-resolution image sequence as $\{\underline{\mathbf{Y}}_k\}$, $N_1 \times N_2$ pixels, to represent the measured data. A original high-resolution image $\underline{\mathbf{X}}$, $qN_1 \times qN_2$ pixels, is to be estimated from the LR sequences, where q is an integer-valued interpolation factor in both the horizontal and vertical directions. To reduce the computational complexity, each frame is separated into overlapping blocks (the shadow blocks as shown in Fig. 1(a) and Fig. 1(b).

For convenience of notation, all overlapping blocked frames will be presented as vector, ordered column-wise lexicographically. Namely, the overlapping blocked LR frame is $\underline{Y}_k \in \mathbb{R}^{M^2}$ ($M^2 \times 1$) and the overlapping blocked HR frame is $\underline{X} \in \mathbb{R}^{q^2M^2}$ ($L^2 \times 1$ or $q^2M^2 \times 1$). Assume that the two images are related via the following equation

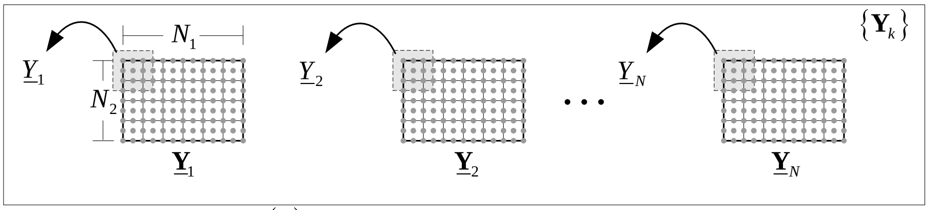
$$\underline{Y}_k = D_k H_k F_k \underline{X} + \underline{V}_k \quad ; k = 1, 2, \ldots, N \qquad (1)$$

where:

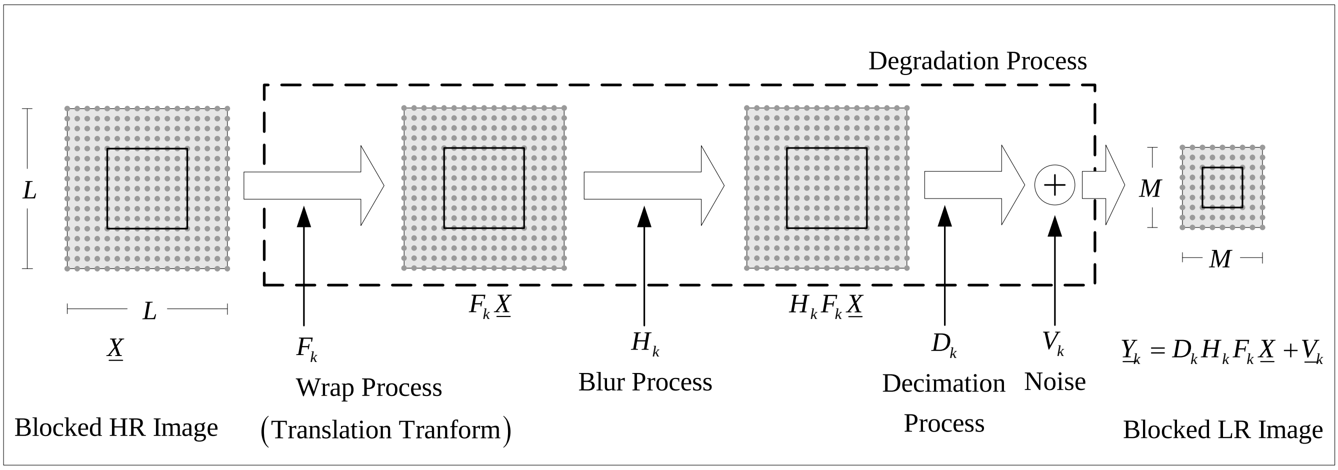
- $\underline{X}$ (vector format) is the original high-resolution blocked image.
- $\underline{Y}_k(t)$ (vector format) is the blurred, decimated, down sampled and noisy blocked image.
- F_k ($F \in \mathbb{R}^{q^2M^2 \times q^2M^2}$, matrix format) stands for the geometric warp (typically, translational motion) between the images $\underline{X}$ and $\underline{Y}_k$.
- H_k ($H_k \in \mathbb{R}^{q^2M^2 \times q^2M^2}$, matrix format) is the blur matrix which is space and time invariant.
- D_k ($D_k \in \mathbb{R}^{M^2 \times q^2M^2}$, matrix format) is the decimation matrix assumed constant.
- $\underline{V}_k$ ($\underline{V}_k \in \mathbb{R}^{M^2}$, vector format) is a system noise.



(a) High-Resolution Image



(b) Low-Resolution Image Sequence



(c) The Relation between Overlapping Blocked HR Image and Overlapping Blocked LR Image Sequence (SRR Observation Model)

Fig. 1. The classical SRR observation model.

The Classical Regularized ML for SRR Algorithm

A popular family of estimators is the family of ML-type estimators (M estimators) (Elad and Feuer 1999a; Nguyen *et al*. 2001). Rewrite the definition of these estimators in the SRR framework as the following minimization problem:

$$\underline{\hat{X}} = \underset{\underline{X}}{\text{ArgMin}} \left\{ \sum_{k=1}^{N} \rho\left(D_k H_k F_k \underline{X} - \underline{Y}_k\right) \right\} \qquad (2)$$

where $\rho(\cdot)$ is a norm estimation. To minimize Eq. (2), the intensity at each pixel of the expected image must be close to those of the original image.

The SRR described in the previous section is an ill-posed problem (Farsiu *et al*. 2004). For the under-determined cases (i.e., when fewer than required frames are available), there exist an infinite number of solutions which satisfy Eq. (2). The solution for squared and over-determined cases is not stable, which means that small amounts of noise in the measurements will result in large perturbations in the final solution. Therefore, a consideration of regularization in SRR algorithm as a mean for picking a stable solution is very useful, if not necessary. Also, regularization can help the algorithm to remove artifacts from the final answer and improve the rate of convergence.

A regularization term compensates the missing measurement information with some general *a priori* information about the desirable HR solution, and is usually implemented as a penalty factor in the generalized minimization cost function. Unfortunately, certain types of regularization cost functions work efficiently for some special types of images but are not suitable for general images.

L2 Norm Estimation with Laplacian Regularization for SRR Algorithm: By using L2 norm estimation (Elad and Feuer 1997; Elad and Feuer 1999a; Elad and Feuer 1999b; Elad and Hecov 2001), the definition of these estimators in the super resolution context is rewritten as the following minimization problem:

$$\underline{X} = \underset{\underline{X}}{\text{ArgMin}} \left\{ \sum_{k=1}^{N} \left(D_k H_k F_k \underline{X} - \underline{Y}_k \right)^2 + \lambda \cdot \left(\Gamma \underline{X} \right)^2 \right\} \quad (3)$$

$$\frac{\partial}{\partial \underline{X}} \left\{ \sum_{k=1}^{N} \left(D_k H_k F_k \underline{X} - \underline{Y}_k \right)^2 + \lambda \cdot \left(\Gamma \underline{X} \right)^2 \right\} = 0$$

$$\sum_{k=1}^{N} \left\{ \frac{\partial}{\partial \underline{X}} \left(D_k H_k F_k \underline{X} - \underline{Y}_k \right)^2 \right\} + \frac{\partial}{\partial \underline{X}} \lambda \cdot \left(\Gamma \underline{X} \right)^2 = 0$$

$$\sum_{k=1}^{N} \left\{ F_k^T H_k^T D_k^T \left(D_k H_k F_k \underline{X} - \underline{Y}_k \right) \right\} + \Gamma^T \cdot \lambda \cdot \left(\Gamma \underline{X} \right) = 0$$

$$\sum_{k=1}^{N} \left\{ F_k^T H_k^T D_k^T \left(D_k H_k F_k \underline{X} - \underline{Y}_k \right) \right\} + \lambda \cdot \left(\Gamma^T \Gamma \underline{X} \right) = 0$$

$$\left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \underline{Y}_k \right) - \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T D_k H_k F_k \right) \underline{X} + \lambda \cdot \left(\Gamma^T \Gamma \right) \underline{X} = 0$$

$$\left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \underline{Y}_k \right) - \begin{pmatrix} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T D_k H_k F_k \right) \\ + \lambda \cdot \left(\Gamma^T \Gamma \right) \end{pmatrix} \underline{X} = 0$$

$$\underbrace{\left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \underline{Y}_k \right)}_{\underline{\mathbf{P}}} - \underbrace{\left(\left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T D_k H_k F_k \right) + \lambda \cdot \left(\Gamma^T \Gamma \right) \right)}_{\underline{\mathbf{R}}} \underline{X} = 0$$

$$\underline{\mathbf{P}} - \underline{\mathbf{R}} \underline{X} = 0$$

By the steepest descent method, the solution of above equation is defined as

$$\underline{\hat{X}}_{n+1} = \underline{\hat{X}}_n + \beta \cdot \left(\underline{\mathbf{P}} - \mathbf{R} \underline{\hat{X}}_n \right)$$

$$\underline{\hat{X}}_{n+1} = \underline{\hat{X}}_n + \beta \cdot \begin{pmatrix} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \underline{Y}_k \right) \\ - \begin{pmatrix} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T D_k H_k F_k \right) \\ + \lambda \cdot \left(\Gamma^T \Gamma \right) \end{pmatrix} \underline{\hat{X}}_n \end{pmatrix}$$

$$\underline{\hat{X}}_{n+1} = \underline{\hat{X}}_n + \beta \cdot \begin{pmatrix} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \underline{Y}_k \right) \\ - \begin{pmatrix} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T D_k H_k F_k \right) \\ + \lambda \cdot \left(\Gamma^T \Gamma \right) \end{pmatrix} \underline{\hat{X}}_n \end{pmatrix}$$

$$\underline{\hat{X}}_{n+1} = \underline{\hat{X}}_n + \beta \cdot \left\{ \begin{matrix} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \left(\underline{Y}_k - D_k H_k F_k \underline{\hat{X}}_n \right) \right) \\ - \left(\lambda \cdot \left(\Gamma^T \Gamma \right) \underline{\hat{X}}_n \right) \end{matrix} \right\} \quad (4)$$

L1 Norm Estimation with Laplacian Regularization for SRR Algorithm: By using L1 norm estimation (Farsiu *et al*. 2004; Farsiu *et al*. 2006a; Farsiu *et al*. 2006b), the definition of these estimators in the super resolution context is rewritten as the following minimization problem:

$$\underline{X} = \underset{\underline{X}}{\text{ArgMin}} \left\{ \sum_{k=1}^{N} \left| D_k H_k F_k \underline{X} - \underline{Y}_k \right| + \lambda \cdot \left(\Gamma \underline{X} \right)^2 \right\} \quad (5)$$

$$\frac{\partial}{\partial \underline{X}} \left\{ \sum_{k=1}^{N} \left\| D_k H_k F_k \underline{X} - \underline{Y}_k \right\|_1 + \lambda \cdot \left(\Gamma \underline{X} \right)^2 \right\} = 0$$

$$\sum_{k=1}^{N} \left\{ \frac{\partial}{\partial \underline{X}} \left\| D_k H_k F_k \underline{X} - \underline{Y}_k \right\|_1 \right\} + \frac{\partial}{\partial \underline{X}} \lambda \cdot \left(\Gamma \underline{X} \right)^2 = 0$$

By the steepest descent method, the solution of Eq. (5) is defined as

$$\underline{\hat{X}}_{n+1} = \underline{\hat{X}}_n + \beta \cdot \left\{ \begin{array}{l} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \operatorname{sign}\left(D_k H_k F_k \underline{\hat{X}}_n - \underline{Y}_k \right) \right) \\ -\left(\lambda \cdot \left(\Gamma^T \Gamma \right) \underline{\hat{X}}_n \right) \end{array} \right\} \quad (6)$$

L2 Norm Estimation with MRF Regularization for SRR Algorithm: By using L2 norm estimation (Schultz and Stevenson 1994), (Schultz and Stevenson 1996), the definition of these estimators in the super resolution context is defined as the following minimization problem:

$$\underline{X} = \underset{\underline{X}}{\operatorname{ArgMin}} \left\{ \sum_{k=1}^{N} \begin{array}{l} \left(\left(D_k \cdot H_k \cdot F_k \cdot \underline{X} - \underline{Y}_k \right)^2 \right) \\ + \left(-\frac{1}{2\beta_{MRF}} \sum_{c \in C} \rho_\alpha \left(\mathbf{d}_c^t \underline{X} \right) \right) \end{array} \right\} \quad (7a)$$

$$\frac{\partial}{\partial \underline{X}} \left\{ \begin{array}{l} \sum_{k=1}^{N} \left\| D_k H_k F_k \underline{X} - \underline{Y}_k \right\|_2^2 \\ + \left(-\frac{1}{2\beta_{MRF}} \sum_{c \in C} \rho_\alpha \left(\mathbf{d}_c^t \underline{X} \right) \right) \end{array} \right\} = 0$$

$$\frac{\partial}{\partial \underline{X}} \left\{ \sum_{k=1}^{N} \left\| D_k H_k F_k \underline{X} - \underline{Y}_k \right\|_2^2 \right\} + \frac{\partial}{\partial \underline{X}} \left(-\frac{1}{2\beta_{MRF}} \sum_{c \in C} \left(\mathbf{d}_c^t \underline{X} \right)^2 \right) = 0$$

$$\frac{\partial}{\partial \underline{X}} \left\{ \sum_{k=1}^{N} \left\| D_k H_k F_k \underline{X} - \underline{Y}_k \right\|_2^2 \right\} + \frac{\partial}{\partial \underline{X}} \left(-\frac{1}{2\beta_{MRF}} \left(\left(\mathbf{d}_1^t \underline{X} \right)^2 + \left(\mathbf{d}_2^t \underline{X} \right)^2 + \left(\mathbf{d}_3^t \underline{X} \right)^2 + \left(\mathbf{d}_4^t \underline{X} \right)^2 \right) \right) = 0$$

$$\frac{\partial}{\partial \underline{X}} \left\{ \sum_{k=1}^{N} \left\| D_k H_k F_k \underline{X} - \underline{Y}_k \right\|_2^2 \right\} + \left(-\frac{1}{\beta_{MRF}} \left(\left(\mathbf{d}_1^{tT} \mathbf{d}_1^t \underline{X} \right) + \left(\mathbf{d}_2^{tT} \mathbf{d}_2^t \underline{X} \right) + \left(\mathbf{d}_3^{tT} \mathbf{d}_3^t \underline{X} \right) + \left(\mathbf{d}_4^{tT} \mathbf{d}_4^t \underline{X} \right) \right) \right) = 0$$

$$\frac{\partial}{\partial \underline{X}} \left\{ \sum_{k=1}^{N} \left\| D_k H_k F_k \underline{X} - \underline{Y}_k \right\|_2^2 \right\} + \left(-\frac{1}{\beta_{MRF}} \left(\mathbf{d}_1^{tT} \mathbf{d}_1^t + \mathbf{d}_2^{tT} \mathbf{d}_2^t + \mathbf{d}_3^{tT} \mathbf{d}_3^t + \mathbf{d}_4^{tT} \mathbf{d}_4^t \right) \underline{X} \right) = 0 \quad (7b)$$

By the steepest descent method, the solution of Eq. (7) is defined as

$$\underline{\hat{X}}_{n+1} = \underline{\hat{X}}_n + \beta \cdot \left\{ \begin{array}{l} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \left(\underline{Y}_k - D_k H_k F_k \underline{\hat{X}}_n \right) \right) \\ -\left(-\frac{1}{\beta_{MRF}} \left(\begin{array}{l} \mathbf{d}_1^{tT} \mathbf{d}_1^t + \mathbf{d}_2^{tT} \mathbf{d}_2^t \\ + \mathbf{d}_3^{tT} \mathbf{d}_3^t + \mathbf{d}_4^{tT} \mathbf{d}_4^t \end{array} \right) \underline{X} \right) \end{array} \right\}$$

$$\underline{\hat{X}}_{n+1} = \underline{\hat{X}}_n + \beta \cdot \left\{ \begin{array}{l} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \left(\underline{Y}_k - D_k H_k F_k \underline{\hat{X}}_n \right) \right) \\ +\left(\frac{1}{\beta_{MRF}} \left(\begin{array}{l} \mathbf{d}_1^{tT} \mathbf{d}_1^t + \mathbf{d}_2^{tT} \mathbf{d}_2^t \\ + \mathbf{d}_3^{tT} \mathbf{d}_3^t + \mathbf{d}_4^{tT} \mathbf{d}_4^t \end{array} \right) \underline{X} \right) \end{array} \right\}$$

$$\underline{\hat{X}}_{n+1} = \underline{\hat{X}}_n + \beta \cdot \left\{ \begin{array}{l} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \left(\underline{Y}_k - D_k H_k F_k \underline{\hat{X}}_n \right) \right) \\ -\left(\lambda \cdot \sum_{c \in C} \rho'_\alpha \left(\mathbf{d}_c^t \underline{\hat{X}}_n \right) \right) \end{array} \right\} \quad (8)$$

where $\rho'_\alpha(\cdot)$ is defined as

$$\rho'_\alpha(\cdot) = 2x \quad ;if \ \rho_\alpha(\cdot) \text{ is a quadratic function} \quad (9a)$$

$$\rho'_\alpha(\cdot) = \begin{cases} 2x & ;|x| \le T_{HUBER} \\ 2T_{HUBER} \cdot \operatorname{sign}(x) & ;|x| > T_{HUBER} \end{cases} \quad (9b)$$

$;if \ \rho_\alpha(\cdot)$ is a Huber function

L1 Norm Estimation with MRF Regularization for SRR Algorithm: By using L1 norm estimation (Farsiu *et al*. 2004; Farsiu *et al*. 2006a; Farsiu *et al*. 2006b), the definition of these estimators in the super resolution context is defined as the following minimization problem:

$$\underline{X} = \underset{\underline{X}}{\operatorname{ArgMin}} \left\{ \begin{array}{l} \sum_{k=1}^{N} \left| D_k \cdot H_k \cdot F_k \cdot \underline{X} - \underline{Y}_k \right| \\ + \left(-\frac{1}{2\beta_{MRF}} \sum_{c \in C} \rho_\alpha \left(\mathbf{d}_c^t \underline{X} \right) \right) \end{array} \right\} \quad (10)$$

$$\frac{\partial}{\partial \underline{X}} \left\{ \begin{array}{l} \sum_{k=1}^{N} \left\| D_k H_k F_k \underline{X} - \underline{Y}_k \right\|_1 \\ + \left(-\frac{1}{2\beta_{MRF}} \sum_{c \in C} \rho_\alpha \left(\mathbf{d}_c^t \underline{X} \right) \right) \end{array} \right\} = 0$$

$$\frac{\partial}{\partial \underline{X}}\left\{\sum_{k=1}^{N}\left\|D_k H_k F_k \underline{X}-\underline{Y}_k\right\|_1\right\} + \frac{\partial}{\partial \underline{X}}\left(-\frac{1}{2\beta_{MRF}}\sum_{c\in C}\left(\mathbf{d}_c^t \underline{X}\right)^2\right)=0$$

$$\frac{\partial}{\partial \underline{X}}\left\{\sum_{k=1}^{N}\left\|D_k H_k F_k \underline{X}-\underline{Y}_k\right\|_1\right\} + \frac{\partial}{\partial \underline{X}}\left(-\frac{1}{2\beta_{MRF}}\begin{pmatrix}\left(\mathbf{d}_1^t \underline{X}\right)^2+\left(\mathbf{d}_2^t \underline{X}\right)^2\\ +\left(\mathbf{d}_3^t \underline{X}\right)^2+\left(\mathbf{d}_4^t \underline{X}\right)^2\end{pmatrix}\right)=0$$

$$\frac{\partial}{\partial \underline{X}}\left\{\sum_{k=1}^{N}\left\|D_k H_k F_k \underline{X}-\underline{Y}_k\right\|_1\right\} + \left(-\frac{1}{\beta_{MRF}}\begin{pmatrix}\left(\mathbf{d}_1^{tT}\mathbf{d}_1^t \underline{X}\right)+\left(\mathbf{d}_2^{tT}\mathbf{d}_2^t \underline{X}\right)\\ +\left(\mathbf{d}_3^{tT}\mathbf{d}_3^t \underline{X}\right)+\left(\mathbf{d}_4^{tT}\mathbf{d}_4^t \underline{X}\right)\end{pmatrix}\right)=0$$

$$\frac{\partial}{\partial \underline{X}}\left\{\sum_{k=1}^{N}\left\|D_k H_k F_k \underline{X}-\underline{Y}_k\right\|_1\right\} + \left(-\frac{1}{\beta_{MRF}}\begin{pmatrix}\mathbf{d}_1^{tT}\mathbf{d}_1^t+\mathbf{d}_2^{tT}\mathbf{d}_2^t\\ +\mathbf{d}_3^{tT}\mathbf{d}_3^t+\mathbf{d}_4^{tT}\mathbf{d}_4^t\end{pmatrix}\underline{X}\right)=0$$

By the steepest descent method, the solution of Eq. (10) is defined as

$$\hat{\underline{X}}_{n+1}=\hat{\underline{X}}_n+\beta\cdot\left\{\begin{array}{l}\left(\sum_{k=1}^{N}F_k^T H_k^T D_k^T\,\mathrm{sign}\left(D_k H_k F_k \hat{\underline{X}}_n-\underline{Y}_k\right)\right)\\ -\left(-\frac{1}{\beta_{MRF}}\begin{pmatrix}\mathbf{d}_1^{tT}\mathbf{d}_1^t+\mathbf{d}_2^{tT}\mathbf{d}_2^t\\ +\mathbf{d}_3^{tT}\mathbf{d}_3^t+\mathbf{d}_4^{tT}\mathbf{d}_4^t\end{pmatrix}\underline{X}\right)\end{array}\right\}$$

$$\hat{\underline{X}}_{n+1}=\hat{\underline{X}}_n+\beta\cdot\left\{\begin{array}{l}\left(\sum_{k=1}^{N}F_k^T H_k^T D_k^T\,\mathrm{sign}\left(D_k H_k F_k \hat{\underline{X}}_n-\underline{Y}_k\right)\right)\\ +\left(\frac{1}{\beta_{MRF}}\begin{pmatrix}\mathbf{d}_1^{tT}\mathbf{d}_1^t+\mathbf{d}_2^{tT}\mathbf{d}_2^t\\ +\mathbf{d}_3^{tT}\mathbf{d}_3^t+\mathbf{d}_4^{tT}\mathbf{d}_4^t\end{pmatrix}\underline{X}\right)\end{array}\right\}$$

$$\hat{\underline{X}}_{n+1}=\hat{\underline{X}}_n+\beta\cdot\left\{\begin{array}{l}\left(\sum_{k=1}^{N}F_k^T H_k^T D_k^T\,\mathrm{sign}\left(D_k H_k F_k \hat{\underline{X}}_n-\underline{Y}_k\right)\right)\\ -\left(\lambda\cdot\sum_{c\in C}\rho'_\alpha\left(\mathbf{d}_c^t \hat{\underline{X}}_n\right)\right)\end{array}\right\} \quad (11)$$

L2 Norm Estimation with BTV Regularization for SRR Algorithm: By using L2 norm estimation (Patanavijit and Jitapunkul 2006), the definition of these estimators in the super resolution context is defined as the following minimization problem:

$$\underline{X}=\underset{\underline{X}}{\mathrm{ArgMin}}\left\{\begin{array}{l}\sum_{k=1}^{N}\left(\left(D_k\cdot H_k\cdot F_k\cdot\underline{X}-\underline{Y}_k\right)^2\right)\\ +\lambda\left(\sum_{l=-P}^{P}\sum_{m=0}^{P}\alpha^{|m|+|l|}\left\|\underline{X}-S_x^l S_y^m\underline{X}\right\|\right)\end{array}\right\} \quad (12)$$

where matrices (operators), S_x^l and S_y^m shift $\mathbf{X}$ by l and m pixels in horizontal and vertical directions, respectively, presenting several scales of derivatives. The scalar weight, α, $0<\alpha<1$, is applied to give a spatially decaying effect to the summation of the regularization terms (Elad 2002).

By the steepest descent method, the solution of Eq. (12) is defined as

$$\hat{\underline{X}}_{n+1}=\hat{\underline{X}}_n+\beta\cdot\left\{\begin{array}{l}\left(\sum_{k=1}^{N}F_k^T H_k^T D_k^T\left(\underline{Y}_k-D_k H_k F_k\hat{\underline{X}}_n\right)\right)\\ -\lambda\left(\sum_{l=-P}^{P}\sum_{m=0}^{P}\alpha^{|m|+|l|}\left(I-S_x^l S_y^m\right)\cdot\mathrm{sign}\left(\hat{\underline{X}}-S_x^l S_y^m\hat{\underline{X}}\right)\right)\end{array}\right\} \quad (13)$$

L1 Norm Estimation with BTV Regularization for SRR Algorithm: By using L1 norm estimation (Farsiu *et al*. 2004; Farsiu *et al*. 2006a; Farsiu *et al*. 2006b), the definition of these estimators in the super resolution context is defined as the following minimization problem:

$$\underline{X}=\underset{\underline{X}}{\mathrm{ArgMin}}\left\{\begin{array}{l}\sum_{k=1}^{N}\left|D_k\cdot H_k\cdot F_k\cdot\underline{X}-\underline{Y}_k\right|\\ +\lambda\left(\sum_{l=-P}^{P}\sum_{m=0}^{P}\alpha^{|m|+|l|}\left\|\underline{X}-S_x^l S_y^m\underline{X}\right\|\right)\end{array}\right\} \quad (14)$$

By the steepest descent method, the solution of Eq. (14) is defined as

$$\underline{\hat{X}}_{n+1} = \underline{\hat{X}}_n + \beta \cdot \left\{ \begin{array}{l} \left(\sum_{k=1}^{N} F_k^T H_k^T D_k^T \operatorname{sign}\left(D_k H_k F_k \underline{\hat{X}}_n - \underline{Y}_k \right) \right) \\ -\lambda \left(\sum_{l=-P}^{P} \sum_{m=0}^{P} \alpha^{|m|+|l|} \left(I - S_x^l S_y^m \right) \cdot \operatorname{sign}\left(\underline{\hat{X}} - S_x^l S_y^m \underline{\hat{X}} \right) \right) \end{array} \right\} \tag{15}$$

Conclusion

This article aims to review the mathematical concept of SRR technology based on stochastic regularization being used during this decade. First, the inverse problem and ill-posed problem concept are introduced. Later, the mathematical models of SRR algorithm based on classical L1 and L2 norms are comprehensively derived as mathematical ill-posed problems. Moreover, several classical regularization functions such as Laplacian, MRF, Huber-MRF and BTV (Bi-Total Variation) regularization are incorporated in the proposed SRR framework in order to remove artifacts from the final answer and to improve the rate of convergence. Finally, the mathematical solutions of each case are obtained by the classical systematical approach.

From this comprehensive mathematical review of the most popular techniques introduced by SRR research community, it is hoped that this article creates mathematical guideline in this research area as well as motivation to develop the relevant techniques.

Acknowledgement

This paper describes a part of the work on a research project affiliated with Assumption University of Thailand, entitled “Video Enhancement Using an Iterative SRR Based on a Robust Stochastic Estimation with an Improved Observation Model,” that has been supported by Research Grant for New Scholar (MRG5180263) from TRF (Thai Research Fund) and CHE (Commission on Higher Education).

References

Borman, S. 2004. Topics in multiframe superresolution restoration. Doctoral Dissertation, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN, USA.

Elad, M.; and Feuer, A. 1997. Restoration of a single superresolution image from several blurred, noisy, and undersampled measured images. IEEE Trans. on Image Processing 6(12): 1646-58.

Elad, M.; and Feuer, A. 1999. Superresolution restoration of an image sequence: adaptive filtering approach. IEEE Trans. on Image Processing 8(3): 387-95.

Elad, M.; and Feuer, A. 1999b. Super-resolution reconstruction of image sequences. IEEE Trans. on Pattern Analysis and Machine Intelligence 21(9): 817-34.

Elad, M.; and Hel-Or, Y. 2001. A fast super-resolution reconstruction algorithm for pure translational motion and common space-invariant blur. IEEE Trans. on Image Processing 10(8): 1187-93.

Elad, M. 2002. On the origin of the bilateral filter and ways to improve it. IEEE Trans. on Image Processing 11(10): 1141-51.

Farsiu, S.; Robinson, M.D.; Elad, M.; and Milanfar, P. 2004b. Fast and robust multiframe super resolution. IEEE Trans. on Image Processing 13(10): 1327-44.

Farsiu, S.; Elad, M.; and Milanfar, P. 2006. Multiframe demosaicing and super-resolution of color images. IEEE Trans. on Image Processing 15(1): 141-59.

Farsiu, S.; Elad, M.; and Milanfar, P. 2006b. Video-to-video dynamic super-resolution for grayscale and color sequences. EURASIP Journal on Applied Signal Processing, vol. 2006, article ID 61859, 15 pages.

Gonzalez, R.C.; and Woods, R.E. 1992. Digital image processing. Addison-Wesley Publishing Company, Reading, MA, USA.

Kang, M.G.; and Chaudhuri, S. 2003. Super-resolution image reconstruction. IEEE Signal Processing Magazine 20(3): 19-20.

Ng, M.K.; and Bose, N.K. 2003. Mathematical analysis of super-resolution methodology.

IEEE Signal Processing Magazine 20(3): 62-74.

Nguyen, N.; Milanfar, P.; and Golub, G. 2001. A computationally efficient superresolution image reconstruction algorithm. IEEE Trans. on Image Processing 10(4): 573-83.

Park, S.C.; Park, M.K.; and Kang, M.G. 2003. Super-resolution image reconstruction: a technical overview. IEEE Signal Processing Magazine 20(3): 21-36.

Patanavijit, V.; and Jitapunkul, S. 2006. An iterative super-resolution reconstruction of image sequences using a Bayesian approach and affine block-based registration. Proc. 14th European Signal Processing Conference (EUSIPCO'06), Florence, Italy, 4-8 September 2006, 5 pages. European Association for Signal Processing (EURASIP), Herent, Belgium.

Patanavijit, V. 2009. Super-Resolution Reconstruction and Its Future Research Direction. AU J.T. 12(3): 149-63.

Rajan, D.; Chaudhuri, S.; and Joshi, M.V. 2003. Multi-objective super resolution concepts and examples. IEEE Signal Processing Magazine 20(3): 49-61.

Schultz, R.R.; and Stevenson, R.L. 1994. A Bayesian approach to image expansion for improved definition. IEEE Trans. on Image Processing 3(3): 233-42.

Schultz, R.R.; and Stevenson, R.L. 1996. Extraction of high-resolution frames from video sequences. IEEE Trans. on Image Processing 5(6): 996-1011.