The Application of the Extended Conjugate Gradient Method on the One-Dimensional Energized Wave Equation

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Abstract

This paper computes the optimal control and state of the one-dimensional Energized wave equation using the Extended Conjugate gradient Method (ECGM. We recalled all vital computational issues in the implementation of the ECGM algorithm on the one-dimensional Energized Wave equation in the paper. With these recalls, program codes were derived which gave various numerical optima controls and states. These optimal controls and states were considered as various points in thin rod as our computational problem is a one dimensional space wave problem.

Keywords: *Extended conjugate gradient method, conjugate gradient method, control operator.*

Introduction

We recalled the Energized wave equation as in (Odio *et al.* 1998) and (Waziri, 2004). The implementation of the ECGM algorithm on the Energized Wave equation follows the pattern of the Conjugate Gradient Method (CGM) developed by (Hestenes and Stiefel, 1952). The difference between the ECGM and the CGM is in the construction of the control operator. The ECGM algorithm, as we have rightly stated, was developed by Ibiejugba and Onumanyi (1984). In the sequel, during the course of the implementation of algorithm computational procedures, we shall recall some fundamental useful results already obtained in the previous papers of this series.

Theory

Recall the unconstrained problem of the optimization Energized wave equation in Waziri and Reju (2006a). The unconstrained problem is here reproduced for accessibility and convenience:

$$\min_{z,u} J(z,u) = \min_{z,u} \int_0^1\int_0^1 [z^2(x,t) + u^2(x,t) + \mu\left[\frac{\partial^2 z(x,t)}{\partial t^2} + \frac{\partial z(x,t)}{\partial t} - \frac{\partial^2 z(x,t)}{\partial x^2} - u(x,t)\right]^2 dxdt\,. \qquad (1)$$

The control penalized gradient is defined as

$$J_{u,i}(z,u,\mu) = 2\{\int_0^1\int_0^1 [z^2(x,t) + u^2(x,t)] + \mu\left\|\frac{\partial^2 z(x,t)}{\partial t^2} + \frac{\partial z(x,t)}{\partial t} - \frac{\partial^2 z(x,t)}{\partial x^2} - u(,t)\right\|^2 dtdx\,. \qquad (2)$$

The state penalized gradient is defined as:

$$J_{z,i}(z,u,\mu) = 2\int_0^1\int_0^1 z(x,t)dtdx\,. \qquad (3)$$

The penalized descent direction as defined in Reju (2001) is:

$$p_{u,i}(x,t) = \int_0^1\int_0^1 J_{u,i}(z,u,\mu)dxdt \qquad (4)$$

From Eq. (2), rewrite Eq. (4) as

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$$J_{u,i}(z,u,\mu) = 2xt\{\int_0^1\int_0^1 [z^2(x,t)+u^2(x,t)]+$$

$$\mu\left\|\frac{\partial^2 z(x,t)}{\partial t^2}+\frac{\partial z(x,t)}{\partial t}-\frac{\partial^2 z(x,t)}{\partial x^2}-u(,t)\right\|^2 dtdx\cdot \quad (5)$$

The state penalized unconstraint gradient is:

$$P_{z,i}(x,t) = 2xt\int_0^1\int_0^1 z(x,t)dxdt\,. \quad (6)$$

Recall the Hamiltonian function for which the optimal solutions in Waziri and Reju (2007) were obtained:

$$u(x,t) = [\frac{\lambda_2 - 2\lambda_1}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin \pi nx -$$

$$\frac{1}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin \pi nx]e^{\lambda_1 t} +$$

$$[\frac{1}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin \pi nx -$$

$$\frac{\lambda_1}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin \pi nx]e^{\lambda_2 t}\,, \quad (7)$$

$$z(x,t) = [\frac{\lambda_2\lambda_1 - 2\lambda_1{}^2}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin \pi nx -$$

$$\frac{\lambda_1}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin \pi nx]e^{\lambda_1 t} +$$

$$[\frac{\lambda_2}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin \pi nx -$$

$$\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin \pi nx]e^{\lambda_2 t}\,, \quad (8)$$

$$z_t(x,t) = [\frac{\lambda_2\lambda_1{}^2 - 2\lambda_1{}^3}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin \pi nx$$

$$-\frac{\lambda_1^2}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin \pi nx]e^{\lambda_1 t} +$$

$$[\frac{\lambda_2^2}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin \pi nx$$

$$-\frac{\lambda_1\lambda_2^2}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin \pi nx]e^{\lambda_2 t}\,, \quad (9)$$

$$z_{tt}(x,t) = [\frac{\lambda_2\lambda_1^3 - 2\lambda_1^4}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin \pi nx$$

$$-\frac{\lambda_1^3}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin \pi nx]e^{\lambda_1 t} +$$

$$[\frac{\lambda_2^3}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin \pi nx -$$

$$\frac{\lambda_1\lambda_2^3}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin \pi nx]e^{\lambda_2 t}\,, \quad (10)$$

$$z_{xx}(x,t) = [\frac{\lambda_2\lambda_1 - 2\lambda_1{}^2}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(0)(-i^2\pi^2)\sin \pi nx$$

$$-\frac{\lambda_1}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(-i^2\pi^2)(0)\sin \pi nx]e^{\lambda_1 t} +$$

$$[\frac{\lambda_2}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_{it}(-i^2\pi^2)(0)\sin \pi nx -$$

$$\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\sum_{i=1}^{\infty} u_i(-i^2\pi^2)(0)\sin \pi nx]e^{\lambda_2 t}\,. \quad (11)$$

The ECGM Implementation Algorithm

In the previous section, we developed some basic computational elements that will be useful in the implementation of the ECGM computational algorithm. The format of this implementation algorithm is reproduced hereunder using the CGM pattern. Nonetheless, if compared to the nature of our problem, the following systematic ECGM is an elegant algorithm which is comparatively better then the CGM algorithm. The descent directions for the control and state are:

$$P_{z,p} = -g_{u,p} = -(a + Bu_{u,p})\,, \quad (12)$$

$$P_{z,p} = -g_{z,p} = -(a + Bu_{zup})\,. \quad (13)$$

The n^{th} iterative optimal state and control are given by this set of equations:

$$z_{i+1} = z_i + \alpha_{z,i}P_{z,i}\,,$$

$$u_{i+1} = u_{u,i} + \alpha_{u,i}P_{u,i}\,. \quad (14)$$

The n^{th} gradients for the state and control are obtained from this set of equations:

$$g_{z,i+1} = g_{z,i} + \alpha_{z,i}B_{z,i}P_{zi}\,,$$

$$g_{u,i+1} = g_{u,i} + \alpha_{u,i}B_{u,i}P_{ui}\,. \quad (15)$$

The n^{th} iterative descent directions for the state and control are derived from these two equations,

$$P_{z,i+1} = -g_{z,i+1} + \beta_{z,i}P_{zi}\,,$$

$$P_{u,i+1} = -g_{u,i+1} + \beta_{u,i}P_{ui}\,, \quad (16)$$

where the various step-lengths are defined in this sequential order:

$$\alpha_{u,j} = \frac{< g_{u,i}, g_{u,i} >}{< P_{u,j}, B_{u,j} P_{u,j} >},$$

$$\alpha_{z,j} = \frac{< g_{z,i}, g_{z,i} >}{< P_{z,j}, B_{z,j} P_{z,j} >}, \quad (17)$$

$$\beta_{u,j} = \frac{< g_{u,i+1}, g_{u,i+1} >}{< g_{u,j}, g_{u,j} >},$$

$$\beta_{z,j} = \frac{< g_{z,i+1}, g_{z,i+1} >}{< g_{z,j}, g_{z,j} >}. \quad (18)$$

Before utilizing Eqs. (16) and (17) in the computational procedures, we need to construct the terms $B_{z,i}P_{z,i}$ and $B_{u,i}P_{u,i}$ vectors as derived from the control operator in Waziri and Reju (2006a) and from Eqs. (4) and (6). We construct the $B_{z,i}P_{z,i}$ vector from the control operator elements B_{11} and B_{22} as obtained in Waziri and Reju (2006b) aforementioned in this sequential order:

$$B_{z,i}P_{z,i} = 3P_{z,0}(x,t) - P_{z,0}(x,0) - 3t\frac{\partial P_{z,0}(x,0)}{\partial t} - t\frac{\partial P_{z,0}(x,0)}{\partial t} + \frac{3t^2\partial^2}{x}\frac{\partial^2 P_{z,0}(x,0)}{\partial t^2} + \frac{3t^2\partial^2}{x^2} + \frac{\partial^2 P_{z,0}(x,0)}{\partial t^2} + \frac{t^2}{2}\frac{\partial^3 P_{z,o}(x,0)}{\partial t^3} + (1+\mu)P_{u,i}. \quad (19)$$

Further simplification yields the state vector $B_{z,i}P_{z,i}$ as follows:

$$B_{z,i}P_{z,i} = 2\int_0^{x1}\int_0^t J_{z,o}(z_i,u_i,\mu)dxdt - 3\int_0^{x}\int_0^t J_{z,i}(z_i,u_i,\mu)dxdt - 3t\int_0^{x1}\int_0^t J_{z,o}(z_i,u_i,\mu)dxdt - t\int_0^{x1}\int_0^t J_{u,o}(z_i,u_i,\mu)dxdt + \frac{6t^2}{x}\int_0^{x1}\int_0^t J_{u,o}(z_i,u_i,\mu)dxdt - \frac{3t^2}{x}z_i(x,0) + \frac{3t^2}{x^2}z_i(x,0) + \frac{t^2}{2}\int_0^x J_{u,o}(z_i,u_i,\mu)dx - \frac{2t^2}{3}u_i(x,o) + \frac{5t^2}{6}\int_0^x J_{u,o}(z_i,u_i,\mu)dx + \frac{x^4}{t^3}\int_0^t J_{z,i}(z_i,u_i,\mu)dt. \quad (20)$$

Now, appropriately applying the analytical solutions as given in the previous section, we obtained the general state vector:

$$B_{z,i}P_{z,i} = [2tx + \frac{xt^3}{3} + \frac{2t^5}{x^3}]\int_0^1\int_0^1\sum_{i=1}^{\infty}u_i(0)\sin\pi ixdtdx + 2\mu[t - 2x]\int_0^1\int_0^1[[\frac{\lambda_1\lambda_2 - 2\lambda_1^2}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty}u_i(0)\sin\pi ix - \frac{\lambda_1}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty}u_{it}(0)\sin\pi ix]e^{\lambda_1 t} + [\frac{\lambda_2}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty}u_{it}(0)\sin\pi ix - \frac{\lambda_1}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty}u_i(0)\sin\pi ix]e^{\lambda_2 t} + 2\frac{\lambda_2 - 3\lambda_1}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty}u_i(0)\sin\pi ix. \quad (21)$$

In a characteristic constructional procedure as for the $B_{z,i}P_{z,i}$ vector, the $B_{u,i}P_{u,i}$ vector is obtained from B_{12} and B_{21} to give:

$$B_{u,i}P_{u,i} = 4P_{u,i}(x,0) + \frac{x^4}{t^3}\frac{\partial P_{u,o}(x,0)}{\partial t} - 2P_{z,i}(x,t) + \frac{x^4}{t^2}\frac{\partial P_{u,o}(x,0)}{\partial t} + \frac{x^4}{2t}\frac{\partial^2 P_{u,o}(x,0)}{\partial t^2} + \mu\frac{\partial^2 P_{u,o}(x,0)}{\partial x^2} - \mu\frac{\partial^2 P_{u,i}(x,t)}{\partial t^2} - \mu\frac{\partial P_{u,i}(x,t)}{\partial t}. \quad (22)$$

Further simplification of Eq. (22) yields the control vector $B_{z,i}P_{z,i}$ as follows:

$$B_{u,i}P_{u,i} = [18tx + \frac{3x^3}{3} +$$

$$x^4]\int_0^1\int_0^1\sum_{i=1}^{\infty} u_i(0)\sin\pi ixdtdx + 2\mu[\frac{2x^5}{t^2} -$$

$$4xt + 2(1+\mu)xt]\int_0^1\int_0^1[[\frac{\lambda_1\lambda_2 - 2\lambda_1^2}{\lambda_2 - \lambda_1} *$$

$$\sum_{i=1}^{\infty} u_i(0)\sin\pi ix -$$

$$\frac{1}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin\pi ix]e^{\lambda_1 t} +$$

$$[\frac{1}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin\pi ix -$$

$$\frac{\lambda_1}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin\pi ix]e^{\lambda_2 t} -$$

$$2[\frac{x^4}{2} + \frac{x^2}{t^2}]\int_0^1\int_0^1[[\frac{\lambda_1\lambda_2 - 2\lambda_1^2}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin\pi ix$$

$$-\frac{\lambda_1}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin\pi ix]e^{\lambda_1 t} +$$

$$[\frac{\lambda_2}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty} u_{it}(0)\sin\pi ix -$$

$$\frac{\lambda_2\lambda_1}{\lambda_2 - \lambda_1}\sum_{i=1}^{\infty} u_i(0)\sin\pi ix]e^{\lambda_2 t}. \quad (23)$$

In conjuncture with the derived penalty cost functional $\mu(x,t) \geq 0$ as given in Waziri and Reju (2006b), we can appropriately substitute Eqs. (21) and (23) into the derived ECGM sequential analytical computational algorithm to obtain the desirable optimal controls and states at different points for the one-dimensional optimization energized wave equation.

The Optimal Control and State Outputs

We make the following observations in summarized tabular form under various values of the initial amplitudes and velocities at various one-dimensional points in a straight line-like rod space and tolerance $(\in = 0.0001)$ by the application of program codes. Tables 1 and 2, respectively, give the general summary of the optimal control and state.

Table 1. The optimal control outputs are given at various dimensional points.

Point profiles	Optimal control $u(x,t)$
2	$-2.4838613502098 \times 10^{-3}$
10	$-2.4857096941292 \times 10^{-3}$
20	$-1.110649111191 \times 10^{-3}$
30	$-1.11064417816319 \times 10^{-3}$
40	$-1.1106424500583 \times 10^{-3}$
50	$-1.11064164995 \times 10^{-3}$
60	$-1.11064125665 \times 10^{-3}$

Table 2. The optimal control outputs.

Point profiles, *n*	Optimal State $z(x,t)$
2	$-2.96177714504089 \times 10^{-7}$
10	$-3.37075632986434 \times 10^{-10}$
20	$-9.4025669392153 \times 10^{-12}$
30	$-1.8569500540829 \times 10^{-12}$
40	$-5.8751944507679 \times 10^{-13}$
50	$-2.4064281727935 \times 10^{-13}$
60	$-1.1660491311665154 \times 10^{-12}$

Conclusion

The results obtained for the optimal states and controls in Tables 1 and 2 are self-revealing. We observe that as the dimensional points "*n*" in space increase from *n* = 2 and *n* = 10, the optimal state values are relatively stable with just a negligible local error. Between *n* = 20 and *n* = 60, the optimal control solutions are stable but the state values alternate in values. Hence, within this range we conclude that the optimal control outputs converge more rapidly than the optimal state as "*n*" increases in value.

The optimal solutions give the solutions of the state and control in a thin medium of a wave propagating in one-dimensional space, say a straight rod. The various optimal values for the state and control represent unique optimal outputs in various points in some given one-dimensional rod.

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