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A Critical Study on the Use of the Gaussian Approximation in Optical Code-Division Multiple-Access Networks

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Abstract

The performance evaluation of optical sparse codes in code-division multiple-access (CDMA) networks is considered for the case of multi-access interference (MAI). A comparison with the standard Gaussian approximation (SGA) shows that the said approximation overestimates the performance of the codes and should not be used for small bit error rates (BERs) due to the weak convergence of the central limit theorem (CLT) for most cases of practical interest. It is also shown that the worst case BER is observed for chip-asynchronous optical CDMA networks.

Keywords*: Code-division multiple-access networks, optical sparse codes, Gaussian approximation, central limit theorem, weak convergence, entropy, uniform distribution.*

Introduction

The standard Gaussian approximation (SGA) based on the normal distribution and graphically represented by the bell curve is frequently used for the performance evaluation of *wireless* code-division multiple-access (CDMA) networks. The applicability of this approach is based on some assumptions concerning the probability density function (pdf) of the cross-correlations between different codewords for arbitrary delays. It is assumed that the *bipolar* pseudo-noise (PN) chip sequences to be used for direct sequence (DS) modulation of the information bits are being designed in such a way that the mean value of the said pdf equals zero. The maximum of the pdf is observed for cross-correlation equal to zero demonstrating the good quasi-orthogonal properties of the codewords, and also the pdf is almost symmetric around the mean. This will result in a faster convergence to the bell curve of the *sum distribution* of the normalized (M-1)-fold *continuous* convolution of the pdf distribution due to multi-access interference (MAI) for M simultaneous users in the network. The term convergence is considered in the weak sense that the bit error rates (BERs) obtained as a result of the integration of the tails of the sum pdf and the bell curve for a fixed decision threshold should be within the same order of magnitude and the central limit theorem (CLT) can be used in first approximation with the increase of M. Such a weak convergence can be observed for bit error rates greater than 0.001 when a significant part of the tail of the distribution contributes to the BER. For lower bit rates however, the convergence rapidly degrades and the exact normalized (M-1)-fold convolution of the pdf distribution should be used for the calculation of the BER instead of SGA.

In the chip-synchronous case, the probabilities of the discrete distribution of the cross-correlations between different codewords for chip delays being divisible by the chip duration are taken into consideration. A central clock for chip synchronization of all simultaneous users is required in the network. A normalized (M-1)-fold *discrete* convolution of the discrete distribution takes place for M simultaneous users. The BER is improved for the same decision threshold due to the reduced value of the standard deviation (SD) of the sum discrete distribution but the convergence to the bell curve with respect to the CLT is not improved because a smaller part of the discrete tail of the sum discrete distribution contributes to the BER for a given decision threshold.

The *optical* code-division multiple-access (CDMA) networks are based on *unipolar* chip sequences where the cross-correlations are positive. As a result, the mean value of the distribution (continuous in the chip asynchronous case or discrete in the chip-synchronous case) is greater than zero, the maximum of the distribution is observed for cross-correlation equal to zero, the distribution is monotonically decreasing with the increase of the cross-correlation, i.e., *the distribution is highly asymmetric around the mean*. It should be expected that the convergence to the bell curve of the normalized (M-1)-fold convolution of the distribution is slower than the one observed for the wireless case.

However, it is usually assumed that the CLT can be applied to small BER in both wireless and optical cases, which in turn follows to an incorrect estimation of the MAI for the majority of cases. The good convergence to the bell curve is taken for granted despite the fact that the CLT is a *weak convergence* theorem. The convergence is relatively good around the mean only but not at the tail of the bell curve. As far from the mean is the point of desired convergence, as more simultaneous users in the network should be included for the normalized (M-1)-fold convolution, which in turn increases the BER since the decision threshold is fixed but the sum SD increases with the increase of M. It should also be noted that the number M of simultaneous users that can be used for the practice is in fact quite limited by the length and the orthogonal properties of the chip sequences. The performance evaluation of CDMA codes is displayed on a waterfall curve representing BER versus M. One can obtain a particular point M on the said curve for a desired BER. The overestimation of the convergence follows to a rapid decrease of the BER values on the waterfall curve when using the SGA which does not correspond to the real impact of the MAI to the overall performance of the CDMA network.

Basic Information about CLT

The central limit theorem for the special case of the binomial distribution was discovered by de Moivre about 1733 and appears in de Moivre (1756). In 1809, Gauß presupposed that the errors of observation are identically normally distributed with expectation zero (see Gauß 1809). Around 1810, Laplace showed that under some basic conditions any sum of a considerable number of mutually independent, identically distributed random variables has a normal distribution as an approximation (see Laplace 1812). Cauchy continued the studies in this direction, which followed to the first rigorous proof of the theorem (see Cauchy 1853). The systematic contributions of the Russian mathematician Liapounoff (1900, 1901) are well known and can be considered as a foundation of the modern probability theory. General studies concerning CLT have been performed by Lindeberg (1922), Lévy (1925, 1937), and Feller (1935). The books by Feller (1968, 1971) and Cramer (1970) contain an excellent overview of the probability theory. The history of the CLT and an extensive bibliography on the topic can be found in Fischer (2000).

As it does follow from the analytical proof (Feller 1968, 1971) of the theorem, a convergence to the bell curve can be observed for *infinite* sums of random variables. In practice, it is often assumed that in first approximation the theorem can be used also for *finite* sums of non-Gaussian random variables. However, this approach strongly depends on the distributions to be used and different distributions would follow to drastically different convergence rates. This results in quite different sizes of the finite sums for which the desired convergence should be observed. There are various cases depending on whether the distribution is discrete or continuous, defined over finite or infinite interval, symmetric or asymmetric around the mean, etc.

This study is related mainly to finite sums of distributions that are discrete, finite, asymmetric around the mean, and mono-tonically decreasing.

Some basic notations are introduced below. Let X_1, X_2, …, X_n, be n independent random variables having identical distributions with mean μ, SD σ, and variance σ^2, and express their sum as:

$$X = X_1 + X_2, + \ldots + X_n.$$

Since the random variables are independent, the sum and variance of the *sum distribution X* are:

$$E[X] = n\mu \tag{1}$$

and

$$E[(X - n\mu)^2] = n\sigma^2, \tag{2}$$

respectively.

By subtracting the mean μ and dividing by σ yields a random variable with mean zero and variance 1. For the said sum of random variables, the random variable:

$$Z = (X - n\mu)/(\sigma n^{1/2}),$$

has mean 0 and variance 1. One can divide the numerator and denominator in the above expression by n, the number of random variables in the sum:

$$Z = (\overline{X}_n - \mu)/(\sigma / n^{1/2}) \tag{3}$$

where

$$\overline{X}_n = (X_1 + X_2 + \ldots + X_n)/n \tag{4}$$

The *central limit theorem* states that regardless the distribution of X_i, as the number of terms in the sum, n, becomes large, the distribution of Z tends to a standard normal.

The pdf of the sum normal distribution X is given by the expression

$$f_X(x) = \frac{1}{\sqrt{2n\pi}\sigma}\exp\left(-\frac{(x-n\mu)^2}{2n\sigma^2}\right) \tag{5}$$

The tail probability P_E for a given fixed threshold X_T, is obtained by integrating the tail of the sum distribution

$$P_E(X_T) = \frac{1}{\sqrt{2n\pi}\sigma}\int_{X_T}^{\infty}\exp\left(-\frac{(u-n\mu)^2}{2n\sigma^2}\right)du \tag{6}$$

The complementary error function $erfc(x)$,

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty}\exp(-u^2)du \tag{7}$$

is usually used for convenience

$$P_E(X_T) = \frac{1}{2}erfc\left(\frac{X_T - n\mu}{\sqrt{2n}\sigma}\right) \tag{8}$$

The practical importance of the theorem is that the bell curve can be used as an approximation to sum distributions. In the past, the calculation of the sum probabilities for arbitrary distributions was a time consuming procedure for a large number of trials. By approximating the said distributions with the normal distribution, one could eventually achieve a numerical simplification, if the convergence rate is acceptable.

The Gaussian approximation is an essential tool for the interpretation of measurement results in metrology (see Taylor and Kuyatt 1994) to be used for international comparisons of measurement standards, basic research, applied research and engineering, calibrating client measurement standards, certifying standard reference materials, and generating standard reference data. Taylor and Kuyatt (1994) consider in details the procedure for the calculation of the statistical uncertainty from measurement results by treating the quoted uncertainty as if a normal distribution and choosing a level of confidence (in percent) around the mean. It should be noted that the said procedure does not deal directly with the tail of the normal distribution but with the majority of measurement results observed around the mean, which explains to a certain extent the applicability of the CLT to metrology.

Apparently, the calculation of errors is associated with the tail of the sum distribution X. There exist two main analytical approaches that can be used for the approximation of the said tail. The first one deals with the convergence rate (see Hall 1982) to the bell curve. The second one determines upper bounds for the probability $P\{X \geq X_T\}$. Despite the numerous fundamental studies on the topic, both approaches are seldom used in practice due to the numerical complications when dealing with arbitrary distributions. The precision of the obtained results is also a major concern. It is much easier to obtain the exact results by calculating the sum distribution X for a particular parent distribution when n is relatively small. For completeness, a short introduction to both analytical approaches is presented in the following two sections.

The Berry-Esséen Theorem

The convergence rate of the sum distribution to standard normal distribution is a subject of intensive studies. The first convergence rate estimates in the CLT were obtained by Liapounov (1900, 1901). Berry (1941) and Esséen (1942) independently obtained the so-called Berry-Esséen theorem. A simplified statement of the theorem is shown below:

For a sum of independent and identically distributed (i.i.d.) random variables $\{X_i\}_{i=1}^{n}$,

$$\sup_{y\in\Re}\left|\Pr\left\{\frac{(X_1+...+X_n)/n-\mu}{\sigma/\sqrt{n}}\le y\right\}-\Phi(y)\right| \le C\frac{\rho}{\sigma^3\sqrt{n}} \quad (9)$$

where C is an absolute constant; μ, σ^2, and $E[|X_1 - \mu|^3] = \rho$ are respectively the finite mean, variance, and absolute third moment of the parent distribution, $\sigma^2 > 0$, $\rho < \infty$; $\Phi(.)$ is the unit Gaussian cumulative distribution function (cdf),

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp(-u^2/2)du\,, \quad (10)$$

and

$$\Phi(x)=\frac{1}{2}erfc\left(\frac{x}{\sqrt{2}}\right). \quad (11)$$

Van Beek (1972) sharpened the constant to $C \le 0.7975$. The best result obtained so far belongs to Shiganov (1986), $C \le 0.7655$.

Chernoff Bounds

There are numerous bounds for the probability $P\{X \ge X_T\}$. Popular ones are the bounds obtained by Chernoff (1952). Let X have *moment generating function* $\phi(t) = E[e^{tX}]$. Then, for any threshold $X_T > 0$,

$P\{X \ge X_T\} \le \exp(-tX_T)\phi(t)$, if $t > 0$,
$P\{X \le X_T\} \le \exp(-tX_T)\phi(t)$, if $t < 0$. (12)

Chernoff bounds hold for all t. The best bound can be obtained by choosing t to minimize $\exp(-tX_T)\phi(t)$. Note that in many cases there exist tighter bounds than Chernoff bounds.

Entropy and the Central Limit Theorem

The *information theory* created by Shannon (1948) influenced numerous studies to the proof of the CTL as a weak convergence law with the use of the *entropy* of the parent distribution (see Barron 1986 and Johnson 2004).

It is well known that for a given variance the differential entropy $h(X)$,

$$h(X)=\int_{-\infty}^{\infty}f_X(x)\log_2\left(\frac{1}{f_X(x)}\right)dx \quad (13)$$

reaches its largest value in the *infinite* interval $(-\infty,\infty)$ when the pdf $f_X(x)$ is *normally* distributed.

Also, the differential entropy $h(X)$,

$$h(X)=\int_{a}^{b}f_X(x)\log_2\left(\frac{1}{f_X(x)}\right)dx \quad (14)$$

has its largest value in the *finite* interval (a, b) when the pdf $f_X(x)$ is *uniformly* distributed.

For *finite* discrete distributions, the largest value of the entropy $H(X)$,

$$H(X)=\sum_{l=1}^{L}p_l\log_2\left(\frac{1}{p_l}\right) \quad (15)$$

is obtained for *equiprobable* probabilities, i.e., $p_1 = p_2 = \ldots = p_L$.

Considering the convergence of finite distributions to the CLT in terms of maximum entropy, it should be expected that the *fastest* convergence to the *normal* distribution in the *infinite* interval $(-\infty,\infty)$ should be obtained by performing n-fold convilution, $n\to\infty$, of *finite uniform* random variables. Also, it should be expected that when comparing the convergence rate to the CLT of two finite distributions, the faster convergence is to be observed for the distribution having the larger entropy.

With the decrease of the entropy of the parent distribution, one should expect a slow convergence to the CLT. If $f_X(x)\to 1$ for given x, or $p_l\to 1$ for given l, the remaining values approach zero. Then $\sigma^2\to 0$ and the entropy approaches zero, thus following to the slow convergence. In the limiting case for $\sigma^2 = 0$ (delta function) the CLT does not hold.

Normal Approximation to the Binomial Distribution

If X is obtained from the *binomial distribution* with parameters n and p_1, then

$$X = \sum_{i=1}^{n} X_i \;, \qquad (16)$$

where $X_1, X_2, ..., X_n$ are independent indicator variables (Bernoulli trials) with probability $P(X_j = 0) = p_0$ and $P(X_j = 1) = p_1$ for each j, where $p_0 + p_1 = 1$. If n is large, the binomial distribution can be approximated by the normal distribution with mean np_1 and variance $n\, p_0 p_1$. The rule of thumb for a fast convergence around the mean is that n should be large enough for $np_1 \geq 5$ and $np_0 \geq 5$.

The converegence rate of the tail of the sum distribution can be evaluated numerically. Fig. 1 represents the results of n-fold convolutions of equiprobable Bernoulli trials ($p_1 = 0.5$) for n = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, and 120. The probabilities of the binomial distribution and the pdf of the normal distribution are presented using a linear vertical scale. The bell curves cannot be resolved from the sum distributions in this graph. The linear scale is used frequently to demonstrate the convergence around the mean, however, this is not the appropriate scaling to be used for revealing the behavior of the tail of the sum distribution.

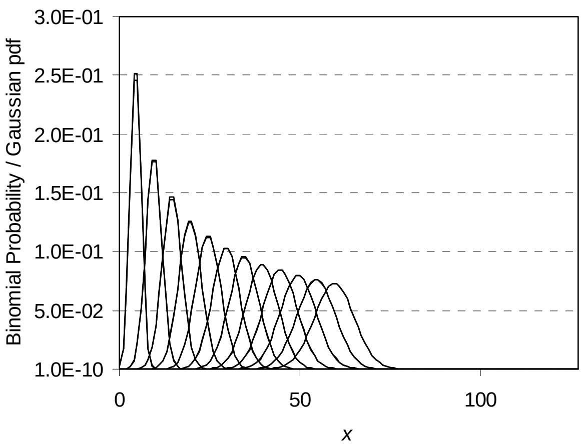

Fig. 1. Normal approximation to the binomial distribution (p_0 = 0.5, p_1 = 0.5) for n = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, and 120 (linear vertical scale)

The same data are also shown in Fig. 2 where the linear vertical scale is replaced by a logarithmic one. Certain small differences not too far from the mean can be found comparing the bell curve and the sum distribution. Obviously, the said differences decrease with the increase of n in accordance with the CLT. It can be concluded that a very good convergence for small n is demonstrated for equiprobable trials.

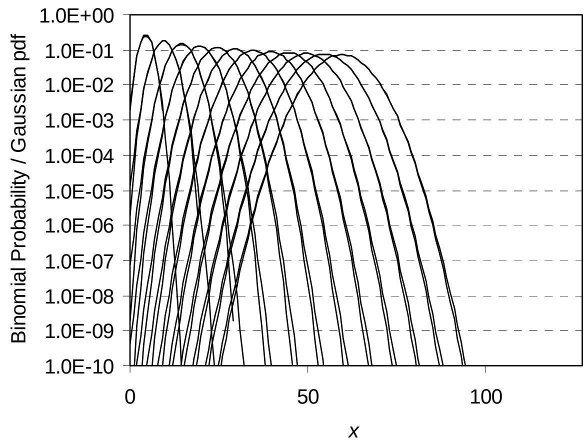

Fig. 2. Normal approximation to the binomial distribution (p_0 = 0.5, p_1 = 0.5) for n = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, and 120 (logarithmic vertical scale)

The tail probability $P\{X \geq X_T\}$ as a function of the sum index n is shown in Fig. 3 for several thresholds T_X. It decreases with the increase of the threshold as expected.

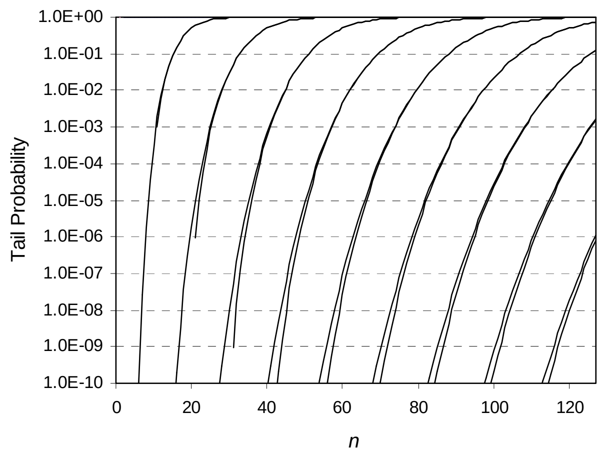

Fig. 3. Tail probability of the normal approximation to the binomial distribution for thresholds T_X = 10, 20, 30, 40, 50, 60, 70, 80, and 90 (p_0 = 0.5, p_1 = 0.5)

The very good convergence demonstrated for p_1 = 0.5 is also related to the maximum entropy of the discrete parent indicator variables. The decrease of the entropy would follow to a slower convergence. Choosing p_0 = 0.95 and p_1 = 0.05, when the probabilities differ by one order of magnitude and the entropy significantly decreases, one can perform an alternative numerical experiment.

Fig. 4 represents the results of n-fold convolutions of such non-equiprobable Bernoulli trials for n = 10, 30, 60, and 120. The mean value of the sum distribution is much smaller that the one from the previous case and the number of displayed curves is reduced here due to the significant overlapping of distributions in a small area. The normal curves and the sum distribution curves do not coincide well around the mean. This is clearly seen even when using a standard linear scale for the vertical axis.

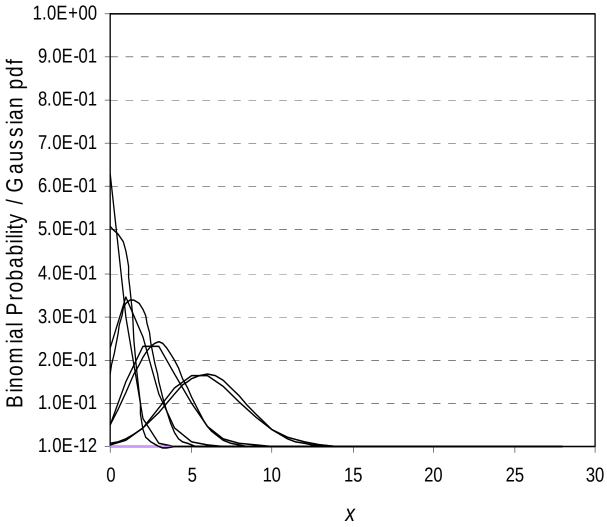

Fig. 4. Normal approximation to the binomial distribution (p_0 = 0.95, p_1 = 0.05) for n = 10, 30, 60, and 120 (linear vertical scale)

The use of a logarithmic vertical scale in Fig. 5 reveals that the differences rapidly increase with the increase of the distance from the mean. Even for n = 120, the curves do not coincide for probability values lower than 0.001. The convergence rate degrades with the decrease of p_1 and similar numerical experiments for lower entropies would only show a slower convergence to the normal distribution.

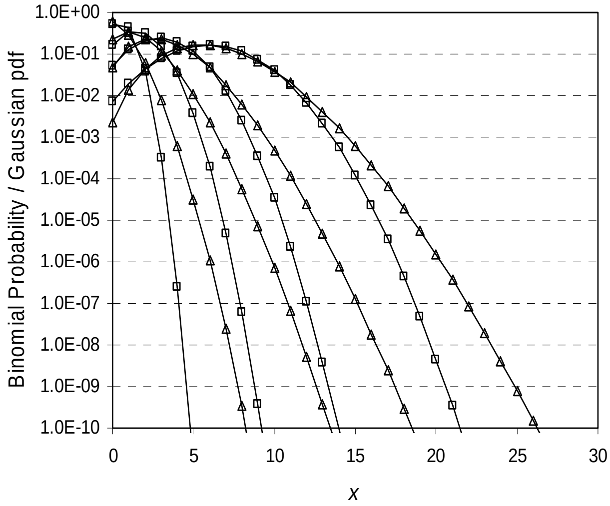

Fig. 5. Normal approximation to the binomial distribution (p_0 = 0.95, p_1 = 0.95) for n = 10, 30, 60, and 120 (logarithmic vertical scale), where the curves with boxes represent the normal distribution and the curves with triangles represent the binomial distribution

The tail probability $P\{X \geq X_T\}$ as a function of the sum index n is shown in Fig. 6 for two chosen thresholds which can be displayed within the range of tail probabilities $10^{-3} \div 10^{-10}$. This is the range being most important for telecommunications applications including CDMA applications as well. It is clearly seen that no convergence is observed for tail probabilities lower that 0.001 due to the very big vertical distance between the curves.

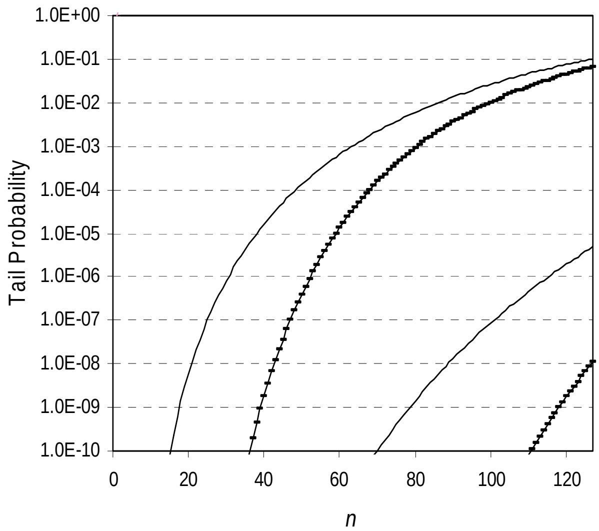

Fig. 6. Tail probability of the normal approximation to the binomial distribution for T_X = 10 and 20 (p_0 = 0.95, p_1 = 0.05), where the lower curves with dashes represent the normal distribution and the upper curves represent the binomial distribution

Optical Chip-Synchronous CDMA and Multinomial Distributions

The results obtained in the previous section can be used as a basis for the analysis of the MAI in non-coherent optical CDMA networks. The chip-synchronous mode of operation will be considered first as it deals with a discrete multinomial distribution of the cross-correlation probabilities and is frequently used by other authors due to the computational simplification in evaluating discrete time delays.

The use of the Gaussian approximation in wireless CDMA networks has been studied by Mowbray and Grant (1992), Rappaport (1996), as well as by many other authors. Modifications of SGA, such as the improved Gaussian approximation (IGA) and the simplified expression for the IGA (SEIGA) can be found in Rappaport (1996). Such modifications are needed due to the weak convergence of the CLT for BER lower than .001 despite the better cross-correlation properties of the bipolar wireless codes when compared to the optical ones.

Yang and Kwong (1995) have been trying to apply the Gaussian approximation from the wireless CDMA case to the optical scenario by assuming zero mean and evaluating the SD of the cross-correlation probability distribution for prime codes and modifications of prime codes.

As it has been pointed out by Lee and Green (1998), the unipolar optical codes have a mean greater than zero in contrast to the bipolar codes used for wireless communications. Mean value greater than zero has been used earlier by Azizoğlu, *et al.* (1992).

The aforesaid difference between bipolar and unipolar codes follows to the increase of the mean of the sum distribution with the increase of the number of users. Given a fixed decision threshold, usually equal to the weight w of the code sequences (perfect power control is assumed), the tail of the sum distribution increases with the increase of both the mean and the SD of the said distribution. This obviously increases the BER. A solution to this problem can be found in the use of *sparse codes*, for which the number of zeroes in the code sequence is much greater that the number of ones. Then the majority of the possible cross-correlations would also be zero, resulting in a sharp maximum of the cross-correlation distribution at the zero cross-correlation point and reduced values of the probabilities for the remaining nonzero cross-correlations. The mean value approaches zero and the SD is minimized. The tail of the sum distribution would remain relatively small with the increase of the number of simultaneous users in the network.

A bit error in an optical CDMA network would occur when the MAI contributes to the accumulation of a total number of optical chip pulses greater than the threshold T_X. The accumulation of optical pulses is counted only at the chip positions set to logical one in the desired user's codeword. The number of said chip positions is equal to the weight w of the code. The BER P_E is calculated as the sum of probabilities for numbers of accumulated optical pulses exceeding the fixed threshold T_X,

$$P_E = \frac{1}{2} \sum_{\substack{l_1+2l_2+...+\lambda_c l_{\lambda_c} \geq T_X \\ l_1+l_2+...+\lambda_c < M}} \Pr(l_1, l_2, ..., l_{\lambda_c}) \qquad (17)$$

where the inclusion of the coefficient 0.5 is based on the assumption that equiprobable data bits (logical 0's and 1's) are being transmitted, and M is the number of simultaneous users in the chip-synchronous optical network. In soft-limiting (no-hard-limiting) mode of operation of the optical receiver, the total number of interfering optical pulses is equal to the sum $l_1 + 2l_2 + ... + \lambda_c l_{\lambda_c}$, where l_i, i = 1, 2, 3, …, λ_c, is the number of simultaneous users interfering at i chip positions [see Azizoğlu *et al.* (1992)].

The parent distribution consists of (1+λ_c) discrete random variables with cross-correlation probabilites p_i, i = 0, 1, 2, …, λ_c, and the sum distribution is a multinomial distribution. The normalized (M-1)-fold convolution of the sum distribution can be represented as follows:

$$\Pr(l_0, l_1, l_2, ..., l_{\lambda_c}) = \prod_{i=0}^{\lambda_c} \binom{M-1}{l_i} q_i^{\,l_i} \qquad (18)$$

where

$$q_i = p_i/2,\ i = 1, 2, \ldots,\ \lambda_c \qquad (19)$$

with the factor 0.5 introduced due to the equiprobable binary transmission,

$$l_0 = (M-1) - \sum_{i=1}^{\lambda_c} l_i \quad (20)$$

and

$$q_0 = 1 - \sum_{i=1}^{\lambda_c} q_i \quad (21)$$

The parameter λ_c is usually set to one (binomial distribution) or 2 (trinomial distribution), because as greater is the size of the parent distribution, as greater are the mean and the SD of the sum distribution, thus following to the increase of the tail after the threshold T_X.

Assuming a perfect optical power control, the threshold equals the weight of the code, $T_X = w$. In reality, however, the threshold should be set as lower as possible, $T_X < w$, which would incerase significantly the BER due to the MAI.

The analytical model introduced by Azizoğlu *et al.* (1992) allows one to calculate the exact MAI by using Eq. (17), which represents the (M-1)-fold discrete convolution of the (1+λ_c)-nomial distribution. The case λ_c = 1 for the binomial distribution was considered in the previous section. The only modification here is that the probabilities (except q_0) should be multiplied by the coefficient 0.5 because of the assumption of equiprobable data bits 0 and 1. The basic conclusion that can be drawn for this case is that the Gaussian approximation is not the appropriate analytical tool for parent distributions with low entropy. As the design of good optical codes requires the probability of the zero cross-correlation to reach its maximum possible value, or minimum entropy, there is a clear contradiction with the working conditions for the Gaussian approximation.

If λ_c = 2, 3, …, the design requirements for good codes would require the probabilities for such higher cross-correlations to be reduced as much as possible and the entropy cannot increase sufficiently when compared to the binomial distribution. From another point of view, if the said probabilities are increasing, this would follow to the immediate increase of the mean and the SD of the sum distribution with respect to the threshold T_X.

Worst Case Bit Error Rates in Chip-Asynchronous Optical CDMA Networks

The cross-correlation distributions of chip-asynchronous networks are continuous. The pdf of the *continuous* distribution is constructed by connecting the discrete values of the probabilities of the chip-synchronous *discrete* distribution with lines and the area below the lines is normalized to unity.

For example, the mean of Bernoulli trials (λ_c = 1) equals

$$\mu_{\text{discrete}} = 0\,p_0 + 1\,p_1 = p_1 \quad (22)$$

and the SD is

$$\sigma_{\text{discrete}} = [0^2 p_0 + 1^2 p_1 - \mu^2_{\text{discrete}}]^{1/2} = [p_1(1-p_1)]^{1/2} \quad (23)$$

The normalization condition for the *continuous* distribution having the pdf function

$$f(x) = p_{0c} + x(p_{1c} - p_{0c}), \quad (24)$$

defined by the line connecting points p_{0c} and p_{1c}, for which $p_1/p_0 = p_{1c}/p_{0c}$, $p_0 > 0$, is

$$\int_0^1 [p_{0c} + u(p_{1c} - p_{0c})]du = 1, \quad (25)$$

After the normalization,

$$p_{0c} + p_{1c} = 2, \quad (26)$$

therefore, $p_{0c} = 2\,p_0$ and $p_{1c} = 2\,p_1$.

The mean is

$$\mu_{\text{continuous}} = \int_0^1 f(u)u\,du = (p_{0c} + 2\,p_{1c})/6 = 1/3 + p_{1c}/6 = (1 + p_1)/3, \quad (27)$$

therefore, $\mu_{\text{continuous}} > \mu_{\text{discrete}}$ for small p_1.

The SD is

$$\sigma_{\text{continuous}} = \sqrt{\int_0^1 f(u)u^2 du - \mu^2_{\text{continuous}}} = \{(p_{0c} + 3\,p_{1c})/12 - [(p_{0c} + 2\,p_{1c})/6]^2\}^{1/2} = \{(1 + p_{1c})/6 - [(2 + p_{1c})/6]^2\}^{1/2} = \{(1 + 2\,p_1)/6 - [(1 + p_1)/3]^2\}^{1/2} \quad (28)$$

Solving the quadratic equation,

$$p_1(1 - p_1) = (1 + 2\,p_1)/6 - [(1 + p_1)/3]^2 \quad (29)$$

one can find the point where $\sigma_{\text{continuous}} = \sigma_{\text{discrete}}$. The first solution is p_1 = 0.0669…, and the second one is p_1 = (1 - 0.0669…) = 0.9330… If p_1 < 0.0669…, or p_1 > 0.9330…, then $\sigma_{\text{continuous}} > \sigma_{\text{discrete}}$.

For most sparse optical CDMA codes with $\lambda_c = 1$, the inequality $p_1 < 0.0669\ldots$ is satisfied, because the good codes are a subject to the condition $p_0 >> p_1$.

Similar results can be obtained for codes with $\lambda_c = 2, 3, \ldots$, having in mind that for good codes $p_0 >> p_1 >> p_2 >> \ldots$, and the inequality $\sigma_{\text{continuous}} > \sigma_{\text{discrete}}$ is also satisfied.

For the majority of cases, both the mean and the SD of the *chip-asynchronous* pdf are greater than the mean and the SD of the *chip-synchronous* discrete distribution, and the *chip-asynchronous* case has the *worst case BER* due to the larger tail probability for a given fixed threshold T_X.

Conclusion

The design of unipolar optical CDMA codes is based on the concept that by increasing the number of zero chips the probability of the zero cross-correlation should increase accordingly and the probabilities of non-zero cross-correlations should decrease. In the ideal case, for extremely long sparse sequences one can have $p_0 \to 1$, thus following to $p_i \to 0$, $0 < i \leq \lambda_c$. As a result, the mean μ of the parent distribution of probabilities is reduced. Also, the SD σ is reduced because for good codes the pdf/probability values of the continuous/discrete cross-correlation parent distributions for the corresponding chip-asynchronous/synchronous cases should strongly and monotonically decrease with the increase of the cross-correlation values. As the distance between the sum mean and the given fixed threshold increases, the corresponding BER decreases. For small BER, the tail of the sum distribution does not converge well to the normal distribution for any realistic number of simultaneous users, $M < 200$. In fact, hundreds and even thousands of such additions should be performed before an acceptable convergence to the normal distribution takes place.

Due to the weak convergence of the central limit theorem, BER lower than 0.001 should be calculated by using an exact knowledge of the sum distribution. The worst BER is observed for chip-asynchronous cases.

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