KMITL Sci. Tech. J. Vol.17 No.1 Jan.-Jun. 2017

The Theory of Geodesics on Some Surface of Revolution

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Abstract

We study the properties of the geodesics on a Randers rotational surface of revolution by using Zermelo navigation data (h,W), where h is the induced Riemannian metric on the surface of revolution and W is the rotational wind. We are in special interested in the half-period function that can be computed by similar methods to the Riemannian case. Our result can be applied to find the structure of the cut locus of a Randers rotational 2-sphere of revolution.

Keywords: Randers rotationalsphere, surface of revolution, Zermelo navigation

1. Introduction

The Riemannian geometry is one of the important research topics for differential geometry field. In general, Riemannian geometry has many interested topics to study, but they are almost well known study. So we are interested to do research in something more complicated or general (nearest the problem in real world) more than Riemannian geometry, that is Finsler geometry ([1], [2]). In this paper we will show that Riemannian case is the special case of Finsler case and we use the Randers metric as an examples for the Finsler case. In the case of a Riemannian surface of revolution, one can study the behaviour of geodesic by using Clairaut relation, we can see that if the geodesic is neither a profile curve nor s parallel then it will be tangent to the some parallel. The length between starting point and returning point can be calculate by using half period function.

The aims of studing the half-period function for Randers rotational case is to find the cut locus on Randers rotational surface of revolution. If we can find the exactly form of this function then we can see the behavior of the cut locus. In this paper, we will show how to construct the half period function for Randers rotational surface of revolution.

2. Materials and Methods

2.1 The geometry of Riemannian surface of revolution

We recall the definition of Riemannian geometry.

Definition 2.1 (Local surface) A subset S of $\mathbb{R}^3$ is called a local surface if there exists a C^∞ map φ of a domain D in $\mathbb{R}^2$ into $\mathbb{R}^3$, i.e. $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$, such that

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1. $S = \varphi(D)$
2. φ is injective.
3. the rank of the matrix $\begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix}$ is 2 at each point on D.

Definition 2.2 (2-sphere of revolution) A compact Riemannian manifold (M,h) homeomorphic to a 2-sphere is called a 2-sphere of revolution if M admits a point p such that for any two points q_1, q_2 on M with $d(p,q_1) = d(p,q_2)$, where $d(,)$ denoted the Riemannian distance function, there exists an isometry f on M satisfying $f(q_1) = q_2$ and $f(p) = p$. The point p is called a pole of M. Let (r,θ) denote geodesic polar coordinates around a pole p of (M,h). The Riemannian metric can be expressed as $h = dr^2 + m(r)^2 d\theta^2$ on $M \setminus \{p,q\}$, where q denotes the unique cut point of p, i.e. p, q are called a pair of poles.

From Definition 2.2 we can construct the classical Riemannian surface of revolution by rotating a unit speed smooth curve $x = f(z)$, where $f:[a,b] \to \mathbb{R}$, $a,b \in \mathbb{R}$ and $f(a) = f(b) = 0$, include in the xz plane around the z axis. We will consider the curve f in parametric form

$$f:\begin{cases} x = m(r) \\ z = z(r) \end{cases},$$

where $r \in [a,b]$, $m > 0$ and of the Euclidean unit speed condition, that is

$$(m'(r))^2 + (z'(r))^2 = 1.$$

Then we obtain the surface of revolution

$$M := \varphi(r,\theta) = \left(m(r)\cos\theta, m(r)\sin\theta, z(r)\right),\ r \in [a,b],\ \theta \in [0,2\pi).$$

One can see that the mapping φ is satisfied Definition 2.1.

We recall the Riemannian metric on surface of revolution is

$$ds^2 = dr^2 + m^2(r)d\theta^2,$$

and the geodesic equations of h -unit speed $\gamma(s) := (r(s),\theta(s))$ of (M,h) are

$$\begin{cases} \dfrac{d^2r}{ds^2} - mm'\left(\dfrac{d\theta}{ds}\right)^2 = 0 \\ \dfrac{d^2\theta}{ds^2} + 2\dfrac{m'}{m}\dfrac{dr}{ds}\dfrac{d\theta}{ds} = 0 \end{cases},$$

with the unit speed condition

$$\left(\frac{dr}{ds}\right)^2 + m^2\left(\frac{d\theta}{ds}\right)^2 = 1.$$

Remark 2.3 We can see that every profile curve, i.e. $\gamma(s) := (r(s),\theta_0)$, where θ_0 is constant, is an h -geodesic and parallel, i.e. $\gamma(s) := (r_0,\theta(s))$, where r_0 is constant and $m'(r_0) = 0$, is an h -geodesic.

Theorem 2.4 (Clairaut relation [3]) If $\gamma(s) := (r(s),\theta(s))$ is a geodesic on surface of revolution (M,h) then the angle $\phi(s)$ between tangent vector of $\gamma(s)$ and the profile curve passing through a point $\gamma(s)$ satisfy

$$m(r(s))\sin\phi(s) = \nu,$$

where ν is constant and called Clairaut constant.

Lemma 2.5 The Clairaut constant for any profile curve is vanishes, i.e. $\nu = 0$.

From Clairaut relation, we can see that if $\gamma(s)$ is neither a profile curve nor a parallel, i.e. $\nu \in (0, m(r))$, then for some $t_1 > 0$, $\gamma(t_1)$ will be tangent to the same parallel of $\gamma(0)$, where $\gamma(0)$ is the emanating point of geodesic.

Let us denoted the geodesic that emanating from a point p_0 with Clairaut constant ν by $\gamma_\nu^{p_0}$.

Remark 2.6 We always assume that our 2-sphere of revolution with a pair of poles p, q satisfying the following properties

1. (M, h) is symmetric with respect to the reflection fixing $r = a$, where $2a$ denotes the distance between p and q.

2. The Gaussian curvature G of M is monotone along a profile curve from the point p to the point on $r = a$.

We can find the length between $\gamma_\nu^{p_0}(0)$ and $\gamma_\nu^{p_0}(t_1)$ by using

Lemma 2.7 (Half period function of Riemannian surface of revolution) Let $\gamma_\nu^{p_0}$ be an h -unit speed geodesic, where $p_0 \in \{r = a\}$ and $\nu \in (0, m(a))$, i.e. p_0 is a point on equator and $\gamma_\nu^{p_0}$ is neither meridian nor equator. From Clairaut relation $\gamma_\nu^{p_0}$ must be tangent to the parallel $\xi(\nu)$ and return to the equatorat $\gamma_\nu^{p_0}(t_1)$. The distance from p_0 to $\gamma_\nu^{p_0}(t_1)$ can be computed by

$$H(\nu) := 2\int_{\xi(\nu)}^{a} \frac{\nu}{m(t)\sqrt{m(t)^2 - \nu^2}}\, dt$$

where H is called half period function.

2.2 The geometry of Randers rotational surface of revolution

In this section, we will consider that if there is a wind blow up on our surface of revolution along the parallel, by using Zermelo navigation problem [4], therefore we obtained

Proposition 2.8 (Randers rotational metric [5]). If (M, h) is a surface of revolution whose profile curve is the bounded function $m(r) < \dfrac{1}{\mu}$ and $W = \mu \cdot \dfrac{\partial}{\partial \theta}$ is the breeze on M blowing along parallels, then the Randers metric $(M, F = \alpha + \beta)$ obtained by the Zermelo's navigation process on M is a Finsler metric on M, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$, $\beta = b_i(x) y^i$ are defined in

$$(a_{ij}) = \begin{pmatrix} \dfrac{1}{1-\mu^2 m^2} & 0 \\ 0 & \dfrac{m^2}{(1-\mu^2 m^2)^2} \end{pmatrix}, \; b_i = \begin{pmatrix} 0 \\ \dfrac{\mu m^2}{1-\mu^2 m^2} \end{pmatrix}, i, j = 1, 2.$$

We obtained the flow of the wind $\varphi(s; r(s), \theta(s)) = (r(s), \theta(s) + \mu s)$.

From [6], the geodesic equation of F -unit geodesic $P(s) = (P^1(s), P^2(s))$ is

$$\begin{cases} \dfrac{d^2 P^1}{ds^2} - mm'\left(\dfrac{dP^2}{ds}\right)^2 - \mu mm'\left(\mu - 2\dfrac{dP^2}{ds}\right) = 0 \\ \dfrac{d^2 P^2}{ds^2} - 2\dfrac{m'}{m}\dfrac{dP^1}{ds}\dfrac{dP^2}{ds} - 2\mu\dfrac{m'}{m}\dfrac{dP^1}{ds} = 0 \end{cases}$$

Remark 2.9 If $\gamma(s) := (r(s), \theta(s))$ be a geodesic on (M, h) then we obtained geodesic $P(s)$ for (M, F) constructed as above by $P(s) = \varphi(s, \gamma(s)) = (r(s), \theta(s) + \mu s)$.

3. Results and Discussion

In this section we assume that there is a wind $W := \mu \frac{\partial}{\partial \theta}$ blowing along the parallels on 2-sphere of revolution (M, h), where $\mu \le \frac{1}{\max\{m(r) : r \in [0, 2a]\}}$. Therefore we obtain the Randers rotational 2-sphere of revolution $(M, F = \alpha + \beta)$.

So, we can obtain our main result

Theorem 3.1 (Half period function of Randers rotational 2-sphere of revolution) Let $P_\nu^{p_0}(s) = (r(s), \theta(s) + \mu s)$ be an F-unit speed geodesic obtained by $\varphi(s; \gamma_\nu^{p_0}(s))$, where $\gamma_\nu^{p_0}(s)$ is an h-unit speed geodesic on (M, h), emanating from $p_0 \in \{r = a\}$ and $\nu \in (0, m(a))$ if the direction of $P_\nu^{p_0}(s)$ is along the wind then $P_\nu^{p_0}(s)$ will tangent to parallel $\xi(\nu)$ at $P_\nu^{p_0}(t_1)$ and return to the equator at $P_\nu^{p_0}(t_0)$. The distance from p_0 to $P_\nu^{p_0}(t_0)$ can be computed by

$$H_F^+(\nu) = H(\nu) + \psi(\nu), \tag{3.1}$$

Where $\psi(\nu) := 2\mu(a - \xi(\nu))$.

In the others hand, if the direction of $P_\nu^{p_0}(s)$ is against the wind then the distance is

$$H_F^-(\nu) = H(\nu) - \psi(\nu). \tag{3.2}$$

Proof. In this proof we denoted $\gamma_\nu^{p_0}(s)$ by $\gamma(s)$ and $P_\nu^{p_0}(s)$ by $P(s)$.

Let $\gamma(s) = (r(s), \theta(s))$ be an h-unit speed geodesic, i.e.

$$\left(\frac{dr}{ds}\right)^2 + m^2(r(s))\left(\frac{d\theta}{ds}\right)^2 = 1. \tag{3.3}$$

Multiply (3.3) with $\left(\frac{ds}{d\theta}\right)^2$, we have

$$\left(\frac{dr}{d\theta}\right)^2 + m^2(r(s)) = \left(\frac{ds}{d\theta}\right)^2. \tag{3.4}$$

From Clairaut relation we have

$$\frac{ds}{d\theta} = \frac{m^2(r(s))}{\nu}. \tag{3.5}$$

Therefore (3.4) can be written as

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{m^2(r(s))(m^2(r(s)) - \nu^2)}{\nu^2}, \tag{3.6}$$

or

$$\frac{d\theta}{dr} = \frac{\nu}{m(r(s))\sqrt{m^2(r(s)) - \nu^2}}. \tag{3.7}$$

By integrating (3.7), we get

$$\theta(b)-\theta(a)=\int_{r(a)}^{r(b)}\frac{\nu}{m(r(s))\sqrt{m^2(r(s))-\nu^2}}\,dr\,. \tag{3.8}$$

From Clairaut relation we know that the geodesic $\gamma(s)$ emanating from the point $\gamma(0)$ on the parallel will tangent to other parallel called $\{r=\xi(\nu)\}$ at $\gamma(t_1)$ and after that it will return to the parallel again. We can see that $\theta(t_0)-\theta(0)=2(\theta(t_0)-\theta(t_1))$, i.e.

$$H(\nu):=\theta(t_0)-\theta(0)=2\int_{\xi(\nu)}^{a}\frac{\nu}{m(t)\sqrt{m(t)^2-\nu^2}}dt\,, \tag{3.9}$$

$H(\nu)$ is called h -half period function. Recall that

$$b-a=\int_a^b ds=\int_{r(a)}^{r(b)}\frac{ds}{dr}dr\,. \tag{3.10}$$

From $P(s)=(r(s),\theta(s)+\mu s)$ obtained from $\gamma(s)$, we have

$$\frac{dP^1}{ds}=\frac{dr}{ds}\,,\ \frac{dP^2}{ds}=\frac{d\theta}{ds}+\mu\,, \tag{3.11}$$

and therefore

$$\begin{aligned}\frac{dP^2}{dP^1}&=\frac{dP^2}{ds}\frac{ds}{dP^1}\\ \frac{dP^2}{dr}&=\left(\frac{d\theta}{ds}+\mu\right)\frac{ds}{dr}\,.\\ &=\frac{d\theta}{dr}+\mu\frac{ds}{dr}\end{aligned} \tag{3.12}$$

By integrating (3.12), we get

$$P^2(r(b))-P^2(r(a))=\int_{r(a)}^{r(b)}\left(\frac{d\theta}{dr}+\mu\frac{ds}{dr}\right)dr\,. \tag{3.13}$$

We will consider in the case that $P(s)$ will tangent to the parallel $\{r=\xi(\nu)\}$ at $P(t_1)$ and return to equator at $P(t_0)$, from (3.9) and (3.10) therefore we got (3.1)

$$\begin{aligned}H_F^+(\nu)&=P^2(t_0)-P^2(0)\\ &=2\int_{\xi(\nu)}^{a}\left(\frac{d\theta}{dr}+\mu\frac{ds}{dr}\right)dr\ .\\ &=H(\nu)+2\mu(a-\xi(\nu))\end{aligned}$$

If we consider the geodesic that against the wind $P(s)=(r(s),\theta(s)-\mu s)$, we get (3.2)

$$H_F^-=H(\nu)-2\mu(a-\xi(\nu))\,.$$

Remark 3.2 The function $\psi(\nu)=2\mu(a-\xi(\nu))$ is decreasing function, where $\xi(\nu)\in(0,a)$, and it is increasing, where $\xi(\nu)\in(a,2a)$.

4. Conclusions

Finally, we can find the half period function for Randers rotational 2-sphere of revolution therefore we can see the structure of cut locus on Randers rotational 2-sphere of revolution as in [7].

5. Acknowledgements

This research was supported by Department of Mathematics, Faculty of Science, KMITL. We thank Professor S. Sabau from Tokai University, Sapporo campus, who provide insight and expertise that greatly assisted the research.

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