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Unitary Convolution and Generalized Möbius Function

Pussadee Yangklan*, Pattira Ruengsinsub and Vichian Laohakosol

Department of Mathematics, Faculty of Science, Kasetsart University,
Bangkok 10900, Thailand

Abstract

The concept of unitary convolution in the commutative ring of arithmetic functions, under the two operations of addition and unitary convolution, is investigated. A generalized unitary Möbius function is defined and its basic properties, which extend those corresponding classical ones are derived. Of particular interest are certain generalized unitary Möbius inversion formula and some characterizations of multiplicative functions.

Keywords: arithmetic function, Möbius inversion formula, multiplicative function, unitary convolution

1. Introduction

An arithmetic function [1] is a complex-valued function defined on the set of positive integers. The set of arithmetic functions, $\mathcal{A}$, is a commutative ring under the usual addition defined by

$$(f+g)(n) = f(n) + g(n)$$

and the unitary convolution, $\sqcup$, defined by

$$(f \sqcup g)(n) = \sum_{d \| n} f\left(n/d\right) g(d) = \sum_{d \| n} g\left(n/d\right) f(d),$$

where $d \| n$ denotes the unitary divisor, i.e., those divisors d for which *gcd (d,n/d)* = 1. The identity element under the unitary convolution is the function

$$I(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}$$

which is also the identity under the Dirichlet convolution, $*$, defined by

$$(f * g)(n) = \sum_{d \mid n} f(d) g(n/d).$$

For $f \in \mathcal{A}$ such that $f(1) \neq 0$, its inverse under the unitary convolution exists and is denoted by f^{-1}. Notice that when n is squarefree, we have $(f \sqcup g)(n) = (f * g)(n)$.
Cohen [2] considered a unitary analogue of the Möbius function

$$\overline{\mu}(n) = (-1)^{\omega(n)}, \tag{1.1}$$

where $\omega(n)$ is the number of distinct prime factors of n with $\omega(1) = 0$. Generalizing Cohen's notion, we define a generalized unitary M bius function by

*Corresponding author: Tel.: +66 851421272
E-mail: pussadee_9@hotmail.com

$$\overline{\mu}_\alpha(n) = (-\alpha)^{\omega(n)},$$

where $\alpha \in \mathbb{C} \setminus \{0\}$ and $\overline{\mu}_0 = I$, so that $\overline{\mu}_1 = \overline{\mu}$. Denote by $\mathcal{M}$, the set of multiplicative functions, viz.,

$$M := \left\{ f \in A \setminus \{0\};\ f(mn) = f(m)f(n) \ \text{ whenever } \ \gcd(m,n) = 1 \right\}.$$

Evidently, $\overline{\mu}_\alpha$ is a multiplicative function, and it is easily checked that a unitary convolution of two multiplicative functions is a multiplicative function.
The objectives in this work are

1) to collect basic properties associated with unitary convolution;
2) to prove the generalized unitaryMöbius inversion formula and to derive relations among the generalized unitary Möbius function and multiplicative functions;
3) to derive some characterizations of multiplicative function.

2. Preliminaries

2.1 The work of Cohen in 1960 [2]

Cohen seemed to be the first person who seriously used the concept of unitary convolution to solve arithmetical problems. First, he defined the unitary Euler's totient, $\overline{\varphi}(n)$. For $a, b \in \mathbb{N}$ with $b > 0$, denote by $(a,b)_*$ the greatest divisor of *a* which is a unitary divisor of *b*. When $(a,b)_* = 1$ the integer *a* is said to be *semiprime* to *b*. The unitary Euler's totient is defined as the number of positive integers in $\{1, 2, \ldots, n\}$ that are semiprimeto *n*, i.e.,

$$\overline{\varphi}(n) = \sharp\left\{ k \in \{1, 2, \ldots, n\};\ (k,n)_* = 1 \right\}.$$

Cohen introduced a trigonometric sum $c^*(m,n)$ for the case of unitary divisors, mimicking the Ramanujan's sum $c(m,n)$ in the ordinary case, by

$$c^*(m,n) = \sum_{(x,n)_*=1} e(mx,n)$$

where $e(m,n) = \exp(2\pi i m / n)$. He proved that

$$\sum_{d \| n} c^*(m,d) = \begin{cases} n & \text{if } n \mid m \\ 0 & \text{if } n \nmid m. \end{cases}$$

Furthermore, he defined

$$\overline{\varphi}(n) = c^*(0,n) \ \text{ and } \ \overline{\mu}(n) = c^*(1,n).$$

Cohen gave the following interesting consequences:

$$\sum_{d\|n} \overline{\varphi}(d) = n, \qquad \sum_{d\|n} \overline{\mu}(d) = I(n), \qquad c^*(m,n) = \sum_{d\|n} \overline{\mu}(d)(n/d),$$

$$\overline{\varphi}(n) = \zeta_1 \sqcup \overline{\mu}(n), \qquad \overline{\mu}(n) = (-1)^{\omega(n)}$$

where $\zeta_1(n) = n$, and for $f, g \in \mathcal{A}$, the unitary Möbius inversion holds, namely,

$$f(n) = \sum_{d\|n} g(d) \ \longleftrightarrow \ g(n) = \sum_{d\|n} \overline{\mu}(d) f(n/d).$$

He also established a number of analytical estimates. If $f(n)$ is defined by $f(n) = (g \sqcup h)(n)$ for $g, h \in \mathcal{A}$ and if $g(n)$ is bounded, then for real number $x \geq 2$, we have

$$\sum_{n \leqslant x} f(n) = \frac{x^2}{2} \sum_{n=1}^{\infty} \frac{g(n)\varphi(n)}{n^3} + O(x^2 \log^2 x).$$

2.2 The work of Cohen in 1961 [3]

Let $\xi, \eta \in \mathcal{A}$ denote functions satisfying

$$\sum_{\substack{d\delta=n \\ (d,\delta)=1}} \xi(d)\eta(\delta) = I(n).$$

Cohen proved

$$f(n) = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} \xi(d)g(\delta) \quad \longleftrightarrow \quad g(n) = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} f(d)\eta(\delta),$$

as well as the generalized unitary inversion formula

$$f(n) = \sum_{\substack{d^k\delta=n \\ (d,\delta)=1}} g(\delta) \quad \longleftrightarrow \quad g(n) = \sum_{\substack{d^k\delta=n \\ (d,\delta)=1}} \overline{\mu}(d) f(\delta)$$

where $k \in \mathbb{N}$. Cohen applied the unitary convolution and the generalized unitary inversion formula to treat some asymptotic problems involving the distribution of certain sets of integers.

2.3 The work of Cohen in 1964 [4]

A positive integer n is said to be k-free if it has no prime factor of multiplicity $\geq k$ and k-full if it has no prime factor of multiplicity $< k$. The set of k-full and k-free integers are denoted by Q_k and L_k respectively. Their characteristic functions are denoted by $q_k(n)$ and $l_k(n)$, respectively. Cohen defined the generalized Möbius function, μ_k^*, to be the multiplicative function such that

$$\mu_k^*(p^e) = \begin{cases} -1, & \text{if } 1 \leqslant e < k \\ 0, & \text{if } e \geqslant k \end{cases}$$

for any prime p and $e \in \mathbb{N}$. Cohen proved that for all $k \geq 1$,

$$\sum_{d \| n} \mu_k^*(d) = l_k(n) \quad \longleftrightarrow \quad \sum_{d \| n} \mu_k^*(d) q_k\left(\frac{n}{d}\right) = I(n)$$

and for $f, g \in \mathcal{A}$, the following inversion formula holds

$$g(n) = \sum_{d \| n} f(d) q_k\left(\frac{n}{d}\right) \quad \longleftrightarrow \quad f(n) = \sum_{d \| n} g(d) \mu_k^*\left(\frac{n}{d}\right).$$

Further, Cohen defined $\overline{\varphi}_k^*(n)$ to be the number of integers m in a complete residue system mod n which are *semiprime* to the maximal unitary divisor of n contained in Q_k. He proved

$$\overline{\varphi}_k^*(n) = \sum_{d \| n} d \mu_k^*\left(\frac{n}{d}\right) \quad \longleftrightarrow \quad \sum_{d \| n} \overline{\varphi}_k^*(d) q_k\left(\frac{n}{d}\right) = n,$$

and

$$\sum_{n \leqslant x} \overline{\varphi}_k^*(n) = \frac{x^2}{0} \sum_{n=1}^{\infty} \frac{\overline{\mu}_k(n)\varphi(n)}{n^3} + O(x^2 \log^2 x).$$

2.4 The work of Horadam [5]

Horadam defined generalized integers and $\varphi^{**}(\ell_n)$ as follows: suppose there is given a finite or infinite sequence $\{p\}$ of real numbers (generalized primes) such that $1 < p_1 < p_2 < \ldots$. Form the set $\{\ell\}$ of all possible p-products; these numbers are called generalized integers. Let $\{\ell_n\}$ be the set of generalized integers. Define $\varphi^{**}(\ell_n)$ to be the number of generalized integers contained in $\{\ell_n\}$ which are semiprimeto ℓ_n. Horadam proved:

- $\sum_{d\delta=\ell_n,(d,\delta)=1} \varphi^{**}(d) = n,$ for all $n \in \mathbb{N};$
- if $f(\ell_n),\ h(\ell_n)$ are multiplicative, then the unitary convolution of $g(\ell_n)$ and $h(\ell_n)$ is also multiplicative;
- a unitary inversion formula: if

$$G(\ell_n) = \sum_{\substack{d\delta=\ell_n \\ (d,\delta)=1}} F(d),$$

then

$$F(\ell_n) = \sum_{\substack{d\delta=\ell_n \\ (d,\delta)=1}} \overline{\mu}(d)G(\delta).$$

2.5 The work of Rao [6]

Rao (see also [2]) gave the following extension of Cohen's totient. For positive integers $n,\ m,\ k,$ let $(n,m^k)_k^*$ denote the largest unitary divisor of m^k that divides n and is a k th power. Denote by $\varphi_k^*(m)$ the number of integers n in a complete residue system mod m^k such that $(n,m^k)_k^* = 1.$ Rao proved that

$$\varphi_k^*(m) = \sum_{d\|m} d^k \overline{\mu}(m/d) = \zeta_k \sqcup \overline{\mu}(m) = m^k \prod_{p|n}\left(1 - \frac{1}{p^{k\nu_p(m)}}\right),$$

where $\zeta_k(n) := n^k$ and $\nu_p(m)$ denotes the highest power of the prime p that divides $m.$ This result implies at once that φ_k^* is a multiplicative function.

Let $\overline{d}(n)$ denote the number of unitary divisors of $n,$ and let $\overline{\sigma}_k(n)$ denote the sum of the k th power of the unitary divisors of n (see also [7]), i.e.,

$$\overline{d}(n) = \sum_{d\|n} 1 = 2^{\omega(n)},\ \overline{\sigma_k}(n) = \sum_{d\|n} d^k = \zeta_k \sqcup U(n).$$

Rao also established the identities

$$\overline{\sigma}_k(n) = \sum_{d\|n} \varphi_k^*(n/d)\overline{d}(d),\ \sum_{d_1\|n} \varphi^*(n/d_1)d_1^k = \sum_{d_2\|n} \varphi_k^*(n/d_2)d_2,\ \sum_{d\|n} \overline{\sigma}_{s+k}(d)\varphi_k^*(n/d) = n^k \overline{\sigma}_s(n).$$

2.6 The work of Rearick [8]

Let $\mathcal{P}$ be the set of all $f \in \mathcal{A}$ such that $f(1)$ is a positive real number. Rearick proved that the groups $\{\mathcal{P},*\},\{\mathcal{M},*\},\{\mathcal{P},\sqcup\},\{\mathcal{M},\sqcup\}$ and $\{\mathcal{A},+\}$ are isomorphic.

2.7 The work of Hansen and Swanson [9]

Hansen and Swanson defined the following concepts.

1. The greatest common unitary divisor (gcud) of positive integers $a,\ b$, written as gcud (a,b), is the integer $d \in \mathbb{N}$ such that $d \| a, d \| b,$ and if $c \| a, c \| b,$ then $c \| d.$
2. The least common unitary divisor (lcud) of positive integers $a,\ b,$ written as lcud(a,b), is the integer d defined by $d = \dfrac{ab}{\text{gcud}(a,b)}.$
3. Any positive integers a,b are unitarily relatively prime if gcud $(a,b) = 1.$
4. A positive integer n is a unitary perfect number if the sum of all the unitary divisors of n is $2n$.

2.8 The work of Johnson [10]

Johnson studied the multiplicative properties of $s_k^*(n)$, defined for $f, g \in \mathcal{A}$, by

$$s_k^*(n) = \sum_{d\|(n,k)_*} f(d)g(k/d).$$

He proved that for $f, g \in \mathcal{A}$,

1. $s_{mk}^*(ab) = s_m^*(a)s_k^*(b)$ whenever $(a,k) = (b,m) = 1$;
2. $s_m^*(ab) = s_m^*(a)$ whenever $(b,m) = 1$;
3. $s_{mk}^*(a) = s_m^*(a)g(k)$ whenever $(a,k) = 1$;
4. if $f, g \in \mathcal{M}$, then $s_k^*(n)$ is multiplicative in k for each fixed n.

2.9 The work of Hsu [11]

Hsu extended the Möbius in version formula based on the prime factorization of integers. Let $n = p_1^{x_1} \cdots p_s^{x_s}$ and $d = p_1^{t_1} \cdots p_s^{t_s}$, with p_i being distinct primes, x_i and t_i being nonnegative integers satisfying $0 \le t_i \le x_i$. Replacing $f(n)$ and $g(d)$ in the Möbius inversion formula by $f((x)) = f(x_1, \ldots, x_s)$ and $g((t)) = g(t_1, \ldots, t_s)$, respectively. Hsu proved the following theorems:

Theorem 2.1 For $f, g \in \mathcal{A}$, we have

$$f(x_1, \ldots, x_s) = \sum_{0 \le t_i \le x_i} g(t_1, \ldots, t_s)$$

if and only if

$$g(x_1, \ldots, x_s) = \sum_{0 \le t_i \le x_i} f(x_1 - t_1, \ldots, x_s - t_s)\mu_1(t_1, \ldots, t_s),$$

where

$$\mu_1(t_1, \ldots, t_s) = \begin{cases} (-1)^{t_1 + \cdots + t_s} & \text{if all } t_i \le 1 \\ 0 & \text{if there is a } t_i \ge 2. \end{cases}$$

Hsu also gave a Möbius inversion formula in vector form. Let $(x) - (t) \equiv (x_1, \ldots, x_s)$ with $(x) \equiv (x_1, \ldots, x_s), (t) \equiv (t_1, \ldots, t_s)$ and $(0) \le (t) \le (x)$, i.e., $0 \le t_i \le x_i$ for all i.

Theorem 2.2 For any given $(r) \equiv (r_1, \ldots, r_s)$ with $r_i \in \mathbb{N}$, we have

$$f((x)) = \sum_{(0) \le (t) \le (x)} \mu_{(r)}^{-1}((t))g((x) - (t))$$

if and only if

$$g((x)) = \sum_{(0) \le (t) \le (x)} \mu_{(r)}((t))f((x) - (t))$$

where $\mu_{(r)}((t))$ and $\mu_{(r)}^{-1}((t))$ are defined by

$$\mu_{(r)}((t)) = \prod_{i=1}^{s} \binom{r_i}{t_i}(-1)^{t_i}, \mu_{(r)}^{-1}((t)) = \prod_{i=1}^{s} \binom{t_i + r_i - 1}{r_i - 1}.$$

2.10 The work of Hsu and Wang [12]

In 1998, Hsu and Wang stated that Theorem 2.2 is also true when each positive integer r_i $(i = 0, \ldots, s)$ is replaced by a real number. They defined a generalized Möbius function by

$$\mu_\alpha(n) = \prod_{p|n} \binom{\alpha}{\nu_p(n)} (-1)^{\nu_p(n)}$$

where $\alpha \in \mathbb{C}$. Note that $\mu_1 = \mu, \mu_0 = I$, and μ_α is a multiplicative function.

It is not difficult to verify that $\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta}$ for complex numbers α, β. They stated the following generalized Möbius inversion formula.

Theorem 2.3 For $f, g \in \mathcal{A}, \alpha \in \mathbb{C}$, we have

$$f(n) = \sum_{d|n} g(d)\mu_\alpha\left(\frac{n}{d}\right) \longleftrightarrow g(n) = \sum_{d|n} f(d)\mu_{-\alpha}\left(\frac{n}{d}\right).$$

They also extended the generalized Möbius function by replacing the arguments with some arithmetic functions. For $\alpha \in \mathcal{A}$, define

$$\mu_{(\alpha)}(n) = \prod_{p|n} \binom{\alpha(p)}{\nu_p(n)} (-1)^{\nu_p(n)}. \tag{2.1}$$

Note that $\mu_{(\alpha)} * \mu_{(\beta)}(n) = \mu_{(\alpha+\beta)}(n)$ for arithmetic functions α, β, $\mu_{(0)} = I$ and is a multiplicative function. The Möbius inversion formula still holds when μ_α and $\mu_{-\alpha}$ are replaced by $\mu_{(\alpha)}$ and $\mu_{(-\alpha)}$, respectively. They also considered the Möbius function in two variables. For $\alpha \in \mathcal{A}, z \in \mathbb{C}$,

$$\mu_{(\alpha)}(n,z) = \prod_{p|n} \left(\frac{\alpha(p)}{\alpha(p) + z\nu_p(n)}\right) \binom{\alpha(p) + z\nu_p(n)}{\nu_p(n)} (-1)^{\nu_p(n)}.$$

This reduces to (2.1) when $z = 0$. Note that $(\mu_{(\alpha)} * \mu_{(\beta)})(n,z) = \mu_{(\alpha+\beta)}(n,z)$ for arithmetic functions α, β and $\mu_{(\alpha)}(n,z)$ is a multiplicative function. Theorem 2.3 is also true when $\mu_\alpha(n)$ is replaced by $\mu_{(\alpha)}(n,z)$, i.e., for $f, g \in \mathcal{A}$, we have

$$f(n) = \sum_{d|n} \mu_{(\alpha)}(n/d, z) g(d) \longleftrightarrow g(n) = \sum_{d|n} \mu_{(-\alpha)}(n/d, z) f(d).$$

Moreover, Theorem 2.2 still holds when each positive integer r_i is replaced by an arithmetic function α_i $(i = 1, \ldots, s)$.

Theorem 2.4 If $(\alpha) = (\alpha_1, \ldots, \alpha_s)$ with all $\alpha_i \in \mathcal{A}$ then

$$f((x)) = \sum_{(0) \le (t) \le (x)} \mu_{(\alpha)}((t), z) g((x) - (t))$$

if and only if

$$g((x)) = \sum_{(0) \le (t) \le (x)} \mu_{(-\alpha)}((t), z) f((x) - (t)),$$

where $\mu_{(\alpha)}((t), z)$ is defined by

$$\mu_{(\alpha)}((t), z) = \prod_{i=1}^{s} \left(\frac{\alpha_i}{\alpha_i + zt_i}\right) \binom{\alpha_i + zt_i}{t_i} (-1)^{t_i}. \tag{2.2}$$

and $\mu_{(-\alpha)}((t), z)$ is taken from (2.2) with (α) being replaced by $(-\alpha) \equiv (-\alpha_1, \ldots, -\alpha_s)$.

2.11 The work of Schinzel [13]

In 1998, Schinzelderived an explicit form of the unitary inverse of $f \in \mathcal{A}$, with $f(1) = 1$, as

$$f^{-1}(n) = \sum_{k=1}^{\omega(n)} \sum_{\substack{d_1 \cdots d_k = n \\ (d_i, d_j) = 1, d_i > 1}} (-1)^k \prod_{i=1}^{k} f(d_i) \quad (n > 1), \quad f^{-1}(1) = 1.$$

2.12 The work of Buschman [14]

In 2003, Buschman considered the unitary Möbius function by defining

$$\overline{\mu}_k = \underbrace{\overline{\mu}\sqcup\overline{\mu}\cdots\sqcup\overline{\mu}}_{\text{k factors}} \quad (k\in\mathbb{N}),$$

with $\overline{\mu}_0 = I, \overline{\mu}_{-1} = U, \overline{\mu}_{-k} = \overline{\mu}_{-1}\sqcup\overline{\mu}_{-1}\cdots\sqcup\overline{\mu}_{-1}$ (k factors), and proved that

$$f = g\sqcup\overline{\mu}_{-k} \longleftrightarrow g = f\sqcup\overline{\mu}_k,$$

which is an extension of the unitary Möbius inversion formula. He also derived the identities

$$\overline{\mu}\sqcup\overline{\sigma}_k = \zeta_k, \quad \overline{\varphi}\sqcup U = \overline{\sigma}, \quad \zeta_k\sqcup\overline{\sigma}_m = \zeta_m\sqcup\overline{\sigma}_k.$$

3. Main Results

3.1 Generalized unitary Möbius inversion formula

In this section, we prove two forms of the generalized unitary Möbius inversion formula.

Lemma 3.1 For $\alpha,\beta\in\mathbb{C}$, we have $\overline{\mu}_\alpha\sqcup\overline{\mu}_\beta = \overline{\mu}_{\alpha+\beta}$.

Proof. When $n=1$, we have

$$\sum_{d\|1}\overline{\mu}_\alpha(d)\overline{\mu}_\beta\left(\frac{1}{d}\right) = \overline{\mu}_\alpha(1)\overline{\mu}_\beta(1) = 1 = \overline{\mu}_{\alpha+\beta}(1).$$

In general, when the prime factorization of n is $n = p_1^{a_1}\ldots p_s^{a_s}$, we have

$$\overline{\mu}_\alpha\sqcup\overline{\mu}_\beta(n) = \overline{\mu}_\alpha(1)\overline{\mu}_\beta(p_1^{a_1}\cdots p_s^{a_s}) + \overline{\mu}_\alpha(p_1^{a_1})\overline{\mu}_\beta(p_2^{a_2}\cdots p_s^{a_s}) + \cdots + \overline{\mu}_\alpha(p_1^{a_1}\cdots p_s^{a_s})\overline{\mu}_\beta(1)$$
$$= (-1)^s\left\{\binom{s}{0}\alpha^0\beta^s + \binom{s}{1}\alpha^1\beta^{s-1} + \cdots + \binom{s}{s}\alpha^s\beta^0\right\} = (-1)^s(\alpha+\beta)^s = \overline{\mu}_{\alpha+\beta}(n).$$

Our first main theorem is:

Theorem 3.2 A) The set of all generalized unitary Möbius functions is an abelian group under the unitary convolution with the identity element $I = \overline{\mu}_0$.

B) (Generalized unitary Möbius inversion formula) Let $f,g\in\mathcal{A}$ and $\alpha\in\mathbb{C}\setminus\{0\}$. Then

$$f(n) = \sum_{d\|n} g(d)\overline{\mu}_{-\alpha}\left(\frac{n}{d}\right) \tag{3.1}$$

if and only if

$$g(n) = \sum_{d\|n} f\left(\frac{n}{d}\right)\overline{\mu}_\alpha(d), \tag{3.2}$$

i.e., $f = g\sqcup\overline{\mu}_{-\alpha} \longleftrightarrow g = f\sqcup\overline{\mu}_\alpha$.

Proof. If $f = g\sqcup\overline{\mu}_{-\alpha}$, then Lemma 3.1 shows that

$$f\sqcup\overline{\mu}_\alpha = g\sqcup\overline{\mu}_{-\alpha}\sqcup\overline{\mu}_\alpha = g\sqcup\overline{\mu}_{-\alpha+\alpha} = g\sqcup I = g.$$

Conversely, if $g = f\sqcup\overline{\mu}_\alpha$, then

$$g\sqcup\overline{\mu}_{-\alpha} = f\sqcup\overline{\mu}_\alpha\sqcup\overline{\mu}_{-\alpha} = f\sqcup\overline{\mu}_{\alpha-\alpha} = f\sqcup I = f.$$

Let the prime factorization of n and its divisor d be

$$n = p_1^{x_1}\cdots p_s^{x_s}, \quad d = p_1^{t_1}\cdots p_s^{t_s},$$

where $x_i\in\mathbb{N}$, $0\le t_i\le x_i$ $(i=1,\ldots,s)$, and $p_1 < p_2 < \cdots < p_s$ are primes. Writing

$$(x)\equiv(x_1,\ldots,x_s),\ (t)\equiv(t_1,\ldots,t_s),\ ((x)-(t))\equiv(x_1-t_1,\ldots,x_s-t_s),$$

The generalized unitary Möbius inversion formula becomes

Theorem 3.3 For $(\alpha)\equiv(\alpha_1,\ldots,\alpha_s)\in(\mathbb{C}\setminus\{0\})^s$, we have

$$f((x))=\sum_{(t)_x} g((t))\overline{\mu}_{(-\alpha)}((x)-(t)) \tag{3.3}$$

if and only if

$$g((x))=\sum_{(t)_x} f((x)-(t))\overline{\mu}_{(\alpha)}((t)), \tag{3.4}$$

where $(t)_x=(t_1,\ldots,t_s)\in\{0,x_i\}^s$ and $\overline{\mu}_{(-\alpha)}((x)-(t))=\prod_{i=1}^{s}\alpha_i^{\omega(p_i^{x_i-t_i})}$. Equivalently,

$$f((x))=(g\sqcup\overline{\mu}_{(\alpha)})((x))\longleftrightarrow g((x))=(f\sqcup\overline{\mu}_{(-\alpha)})((x)).$$

To prove Theorem 3.3, we need the following lemma.

Lemma 3.4 For $\alpha\in\mathbb{C}\setminus\{0\}$, $x\in\mathbb{N}$ and $u\in\{0,x\}$, we have

$$\sum_{v\in\{0,x\}}\alpha^{\omega(p^{v-u})}(-\alpha)^{\omega(p^{x-v})}=\begin{cases}1 & if\ u=x\\ 0 & if\ u=0.\end{cases}$$

Proof. The result easily follows from

$$\sum_{v\in\{0,x\}}\alpha^{\omega(p^{v-u})}(-\alpha)^{\omega(p^{x-v})}=\begin{cases}\alpha^{\omega(p^0)} & =1\quad if\ u=x\\ (-\alpha)^{\omega(p^x)}\alpha^{\omega(p^0)}+\alpha^{\omega(p^x)} & =0\quad if\ u=0.\end{cases}$$

We return now to the proof of Theorem 3.3.

If $f((x))=(g\sqcup\overline{\mu}_{(-\alpha)})((x))$, then

$$\begin{aligned}\sum_{(t)_x} f((x)-(t))\overline{\mu}_{(\alpha)}((t)) &= \sum_{t_i\in\{0,x_i\}} f(x_1-t_1,\ldots,x_s-t_s)\,\overline{\mu}_{(\alpha)}(t_1,\ldots,t_s)\\ &=\sum_{t_i\in\{0,x_i\}}\ \sum_{u_i\in\{0,x_i-t_i\}} g(u_1,\ldots,u_s)\,\overline{\mu}_{(\alpha)}(t_1,\ldots,t_s)\overline{\mu}_{(-\alpha)}(x_1-t_1-u_1,\ldots,x_s-t_s-u_s)\\ &=\sum_{u_i\in\{0,x_i\}} g(u_1,\ldots,u_s)\{\sum_{v_1\in\{0,x_1\}}\alpha_1^{\omega(p_1^{v_1-u_1})}(-\alpha_1)^{\omega(p_1^{x_1-v_1})}\}\cdots\{\sum_{v_s\in\{0,x_s\}}\alpha_s^{\omega(p_s^{v_s-u_s})}(-\alpha_s)^{\omega(p_s^{x_s-v_s})}\}\\ &=g(x_1,\ldots,x_s),\end{aligned}$$

by Lemma 3.4.The converse implication is proved by retreating the above steps.

Next, we verify two identities related to generalized unitary Möbius function.

Proposition 3.5 A. Let $\alpha,\beta\in\mathbb{C}$. If $f\in\mathcal{M}$, then

$$(\overline{\mu}_\alpha f\sqcup\overline{\mu}_\beta)(n)=\prod_{p|n}(-\beta-\alpha f(p^{\nu_{p(n)}})).$$

B. For $\alpha\in\mathbb{C},\gamma\in\mathbb{R}$ and $n\in\mathbb{N}$, we have

$$(\overline{\mu}_\alpha\sqcup\zeta_\gamma)(n)=\prod_{p|n}(p^{\gamma\nu_{p(n)}}-\alpha)\qquad(\zeta_\gamma(n):=n^\gamma).$$

Proof. A. For $n=1$, we have

$$(\overline{\mu}_\alpha f\sqcup\overline{\mu}_\beta)(1)=\overline{\mu}_\alpha f(1)\overline{\mu}_\beta(1)=1.$$

Let $g=\overline{\mu}_\alpha f\sqcup\overline{\mu}_\beta$, so that $g\in\mathcal{M}$ .Let $n=p_1^{a_1}\cdots p_s^{a_s}$ be its prime factorization. Clearly,

$$g(p_i^{a_i})=\sum_{d\|p_i^{a_i}}\overline{\mu}_\alpha(d)f(d)\overline{\mu}_\beta\left(\frac{p_i^{a_i}}{d}\right)=\overline{\mu}_\alpha(1)f(1)\overline{\mu}_\beta(p_i^{a_i})+\overline{\mu}_\alpha(p_i^{a_i})f(p_i^{a_i})\overline{\mu}_\beta(1)=-\beta-\alpha f(p_i^{a_i}),$$

and so

$$g(n)=\prod_{i=1}^{s}g(p_i^{a_i})=\prod_{p|n}(-\beta-\alpha f(p^{\nu_p(n)})).$$

B. For $n=1$, we have

$$(\bar{\mu}_\alpha \sqcup \zeta_\gamma)(1) \; = \; \bar{\mu}_\alpha(1)\zeta_\gamma(1) \; = \; 1.$$

Putting $g = \bar{\mu}_\alpha \sqcup \zeta_\gamma \in \mathcal{M}$, and evaluating at prime powers, we get

$$g(p_i^{a_i}) = \sum_{d \| p_i^{a_i}} \zeta_\gamma(d)\bar{\mu}_\alpha\left(\frac{p_i^{a_i}}{d}\right) = \; \zeta_\gamma(1)\bar{\mu}_\alpha(p_i^{a_i}) + \zeta_\gamma(p_i^{a_i})\bar{\mu}_\alpha(1) = \; -\alpha + p_i^{\gamma a_i},$$

yielding

$$g(n) = \prod_{p|n}(p^{\gamma v_p(n)} - \alpha).$$

3.2 Multiplicative function and applications

In this section, characterizations of multiplicative functions using unitary products are derived. We start with an auxiliary result.

Lemma 3.6 For $n \in \mathbb{N}, n > 1$, and $\alpha \in \mathbb{R}$, there are only two values of α (namely 0 and 1) for which the expression $(1-\alpha)^n - 1 - (-\alpha)^n$ can vanish.

Proof. Putting

$$F(\alpha) := (1-\alpha)^n - 1 - (-\alpha)^n,$$

we see that $F(0) = 0$ and

$$F(1) = 0 - 1 - (-1)^n = \begin{cases} 0 & if \; n \; is \; odd \\ -2 & if \; n \; is \; even. \end{cases}$$

The asserted result follows from the following claims.

Claim 3.7 If $\alpha < 0$, then $F(\alpha) > 0$.

Proof of Claim 3.7 If $\alpha < 0$, then

$$F(\alpha) = 1 + \binom{n}{1}(-\alpha) + \binom{n}{2}(-\alpha)^2 + \cdots + \binom{n}{n}(-\alpha)^n - 1 - (-\alpha)^n > 0.$$

Claim 3.8 For $\alpha \in \mathbb{R} \setminus \{0\}$, if $F(\alpha) = 0$, then $F(1/\alpha) = 0$.

Proof of Claim 3.8 If $F(\alpha) = 0$, then

$$F\left(\frac{1}{\alpha}\right) = \left(1 - \frac{1}{\alpha}\right)^n - 1 - \left(-\frac{1}{\alpha}\right)^n = \frac{(-1)^n}{\alpha^n}F(\alpha) = 0.$$

Claim 3.9 If α belongs to the open interval (0,1), then $F(\alpha) \neq 0.$

Proof of Claim 3.9 If $\alpha \in (0,1)$, then

$$F(\alpha) = (1-\alpha)^n - 1 - (-\alpha)^n < (1-\alpha) - 1 - (-\alpha)^n = -\alpha - (-\alpha)^n < 0,$$

because $n > 1$.

We now present our characterizations of multiplicative functions.

Theorem 3.10 Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{A}$.

1. If $f \in \mathcal{M}$, then $\bar{\mu}_\alpha f \sqcup f = \bar{\mu}_{\alpha-1} f$.
2. If $f(1) \neq 0, \alpha \notin \{0,1\}$ and $\bar{\mu}_\alpha f \sqcup f = \bar{\mu}_{\alpha-1} f$, then $f \in \mathcal{M}$.

Proof. 1. If $f \in \mathcal{M}$, then

$$(\bar{\mu}_\alpha f \sqcup f)(n) \; = \sum_{d\|n} \bar{\mu}_\alpha(d) f(d) f\left(\frac{n}{d}\right) = f(n)\sum_{d\|n}\bar{\mu}_\alpha(d) = f(n)\bar{\mu}_{\alpha-1}(n).$$

2. For $n=1$, we have

$$(\bar{\mu}_\alpha f \sqcup f)(1) = \bar{\mu}_{\alpha-1}(1) f(1),$$

using $f(1) \neq 0$ and $\alpha \notin \{0,1\}$. For $n = p_1^{a_1} p_2^{a_2}$ with distinct primes p_1, p_2, from

$$\bar{\mu}_{\alpha-1}(p_1^{a_1} p_2^{a_2}) f(p_1^{a_1} p_2^{a_2}) = \; (\bar{\mu}_\alpha f \sqcup f)(p_1^{a_1} p_2^{a_2})$$

we get

$$(1-\alpha)^2 f(p_1^{a_1} p_2^{a_2}) = f(p_1^{a_1} p_2^{a_2}) + \overline{\mu}_\alpha(p_1^{a_1}) f(p_1^{a_1}) f(p_2^{a_2}) + \overline{\mu}_\alpha(p_2^{a_2}) f(p_2^{a_2}) f(p_1^{a_1}) + \overline{\mu}_\alpha(p_1^{a_1} p_2^{a_2}) f(p_1^{a_1} p_2^{a_2}).$$

Simplifying and using $\alpha \notin \{0,1\}$, we deduce that $f(p_1^{a_1} p_2^{a_2}) = f(p_1^{a_1}) f(p_2^{a_2})$. Assume that

$$f(p_1^{a_1} \cdots p_k^{a_k}) = f(p_1^{a_1}) \cdots f(p_k^{a_k}) \quad (k = 2,3,\ldots, s-1;\ s \geq 3)$$

for distinct primes $p_1,\ldots, p_k$. Let $p_1,\ldots, p_s$ be distinct and $a_1,\ldots, a_s \in \mathbb{N}$. From

$$\overline{\mu}_{\alpha-1} f(p_1^{a_1} \cdots p_s^{a_s}) = \sum_{d \| p_1^{a_1} \cdots p_s^{a_s}} \overline{\mu}_\alpha(d) f(d) f\left(\frac{p_1^{a_1} \cdots p_s^{a_s}}{d}\right),$$

expanding the first and the last terms on the right hand side, we arrive at

$$(1-\alpha)^s f(p_1^{a_1} \cdots p_s^{a_s}) = \overline{\mu}_\alpha f(1) f(p_1^{a_1} \cdots p_s^{a_s}) + \overline{\mu}_\alpha f(p_1^{a_1} \cdots p_s^{a_s}) f(1) + \sum_{\substack{d \| p_1^{a_1} \cdots p_s^{a_s} \\ d \notin \{1,\, p_1^{a_1} \cdots p_s^{a_s}\}}} \overline{\mu}_\alpha f(d) f\left(\frac{p_1^{a_1} \cdots p_s^{a_s}}{d}\right).$$

Using the induction hypothesis and simplifying, we get

$$\left\{(1-\alpha)^s - 1 - (-\alpha)^s\right\} f(p_1^{a_1} \cdots p_s^{a_s}) = \left\{(1-\alpha)^s - 1 - (-\alpha)^s\right\} f(p_1^{a_1}) \cdots f(p_s^{a_s}).$$

Since $\alpha \notin \{0,1\}$, Lemma 3.6.shows that $f(p_1^{a_1} \cdots p_s^{a_s}) = f(p_1^{a_1}) \cdots f(p_s^{a_s})$. Hence f is multiplicative.

The conditions imposed in Theorem 3.10 are analyzed in the next theorem.

Theorem 3.11 Let $f \in \mathcal{A}$, and $\alpha \in \mathbb{C}$.

A. Assume that $f(1) = 0$, and that $\overline{\mu}_\alpha f \sqcup f = \overline{\mu}_{\alpha-1} f$

A1. If $\alpha \neq 1$, then $f \equiv 0$.

A2. If $\alpha = 1$, then there are infinitely many f satisfying $\overline{\mu}_\alpha f \sqcup f = \overline{\mu}_{\alpha-1} f$.

B. If $f(1) = 1$, then $\overline{\mu}_0 f \sqcup f = \overline{\mu}_{-1} f$.

Proof. A1.Evaluating at p^a, $a \in \mathbb{N}$, p prime, we get

$$\overline{\mu}_{\alpha-1}(p^a) f(p^a) = \sum_{d \| p^a} \overline{\mu}_\alpha(d) f(d) f\left(p^a / d\right),$$

which immediately implies $f(p^a) = 0$. Next, evaluating at $n = p_1^{a_1} \cdots p_s^{a_s}$ for distinct primes p_i and $a_i \in \mathbb{N}$, we have

$$\overline{\mu}_{\alpha-1} f(p_1^{a_1} \cdots p_s^{a_s}) = \sum_{d \| p_1^{a_1} \cdots p_s^{a_s}} \overline{\mu}_\alpha(d) f(d) f\left(\frac{p_1^{a_1} \cdots p_s^{a_s}}{d}\right),$$

which by induction shows that $(1-\alpha)^s f(p_1^{a_1} \cdots p_s^{a_s}) = 0$, and so $f(p_1^{a_1} \cdots p_s^{a_s}) = 0$.

A2. Let $n_0 \in \mathbb{N} \setminus \{1\}$, $a \in \mathbb{C} \setminus \{0\}$. Consider $f \in \mathcal{A}$ defined by

$$f(n) = \begin{cases} a & if\ n = n_0 \\ 0 & if\ n \neq n_0. \end{cases}$$

Then $f(1) = 0$. For $n \in \mathbb{N}$, whether n_0 is a unitary divisor of n or not, we find that

$$(\overline{\mu} f \sqcup f)(n) = \sum_{d \| n} \overline{\mu} f(d) f\left(\frac{n}{d}\right) = 0 = \overline{\mu}_0 f(n).$$

Since there are infinitely many such arithmetic functions f, the desired assertion follows.

B. If $f(1) = 1$, then

$$\overline{\mu}_{-1} f = Uf = f = I \sqcup f = If \sqcup f = \overline{\mu}_0 f \sqcup f.$$

Our next application deals with the concept of discriminative products.For g, $h \in \mathcal{A}$, a unitary product $k = g \sqcup h$ is said to be *discriminative* if

$$k(n) = g(n)h(1) + g(1)h(n)$$

holds only when $\omega(n) = 1$.

A unitary product $k = g \sqcup h$ is said to be *semi-discriminative* if

$$k(n) = g(n)h(1) + g(1)h(n)$$

holds only when $\omega(n) = 1$ or 1.

Using these two concepts, we have the following characterizations of multiplicative functions.

Theorem 3.12 Let $f \in \mathcal{A}$.

1. If $f \in \mathcal{M}$, then f distributes over any unitary discriminative product, i.e.,
$$f(g \sqcup h) = fg \sqcup fh$$
for all $g, h \in \mathcal{A}$.
2. If $f(1) \neq 0$ and f distributes over some unitary discriminative product k, then $f \in \mathcal{M}$.
3. If $f(1) = 1$ and f distributes over some unitary semi-discriminative product k, then $f \in \mathcal{M}$.

Proof. 1. Assume that $f \in \mathcal{M}$. Then for all $g, h \in \mathcal{A}$, $n \in \mathbb{N}$, we have

$$f(g \sqcup h)(n) = f(n)\sum_{d\|n} g(d)h\left(\frac{n}{d}\right) = \sum_{d\|n} g(d)f(d)h\left(\frac{n}{d}\right)f\left(\frac{n}{d}\right) = (fg \sqcup fh)(n).$$

2. Assume that $f(1) \neq 0$, $k = g \sqcup h$ is a unitary discriminative product, and $fk = fg \sqcup fh$. For $n = 1$, we have $f(1) = 1$.Next, we show that

$$f(p_1^{a_1} \cdots p_s^{a_s}) = f(p_1^{a_1}) \cdots f(p_s^{a_s}) \tag{3.5}$$

for $s \in \mathbb{N}$, all distinct primes $p_1, \ldots, p_s$, and $a_1, \ldots, a_s \in \mathbb{N}$. Clearly, (3.5) holds for $s = 1$. Let p_1, p_2, be distinct primes and $a_1, a_2 \in \mathbb{N}$. Then

$$fk(p_1^{a_1} p_2^{a_2}) = (fg \sqcup fh)(p_1^{a_1} p_2^{a_2}) = \sum_{d\|p_1^{a_1} p_2^{a_2}} fg(d)\, fh\left(\frac{p_1^{a_1} p_2^{a_2}}{d}\right).$$

Thus,

$$0 = \{f(p_1^{a_1} p_2^{a_2}) - f(p_1^{a_1})f(p_2^{a_2})\} \sum_{\substack{d\|p_1^{a_1} p_2^{a_2} \\ d \notin \{1,\, p_1^{a_1} p_2^{a_2}\}}} g(d)h\left(\frac{p_1^{a_1} p_2^{a_2}}{d}\right)$$
$$= \{f(p_1^{a_1} p_2^{a_2}) - f(p_1^{a_1})f(p_2^{a_2})\}\{k(p_1^{a_1} p_2^{a_2}) - (g(1)h(p_1^{a_1} p_2^{a_2}) + g(p_1^{a_1} p_2^{a_2})h(1))\}.$$

Since k is unitary discriminative, (3.5) holds for $s = 2$.Let $s \geq 3$, and assume that (3.5) holds for all positive integers whose number of distinct prime factors is less than *s*. Let $p_1, \ldots, p_s$ be distinct primes and $a_1, \ldots, a_s \in \mathbb{N}$. Then

$$fk(p_1^{a_1} \cdots p_s^{a_s}) = (fg \sqcup fh)(p_1^{a_1} \cdots p_s^{a_s}) = \sum_{d\|p_1^{a_1} \cdots p_s^{a_s}} fg(d)\, fh\left(\frac{p_1^{a_1} \cdots p_s^{a_s}}{d}\right)$$
$$= f(p_1^{a_1}) \cdots f(p_s^{a_s}) \sum_{\substack{d\|p_1^{a_1} \cdots p_s^{a_s} \\ d \notin \{1,\, p_1^{a_1} \cdots p_s^{a_s}\}}} g(d)h\left(\frac{p_1^{a_1} \cdots p_s^{a_s}}{d}\right)$$
$$+ f(p_1^{a_1} \cdots p_s^{a_s})g(p_1^{a_1} \cdots p_s^{a_s})h(1) + g(1)h(p_1^{a_1} \cdots p_s^{a_s})f(p_1^{a_1} \cdots p_s^{a_s}).$$

Thus,

$$0=\{f(p_1^{a_1}\cdots p_s^{a_s})-f(p_1^{a_1})\cdots f(p_s^{a_s})\}\sum_{\substack{d\parallel p_1^{a_1}\cdots p_s^{a_s}\\ d\notin\{1,\,p_1^{a_1}\cdots p_s^{a_s}\}}} g(d)h\left(\frac{p_1^{a_1}\cdots p_s^{a_s}}{d}\right)$$

$$=\ \{f(p_1^{a_1}\cdots p_s^{a_s})-f(p_1^{a_1})\cdots f(p_s^{a_s})\}\times$$
$$\{(g\sqcup h)(p_1^{a_1}\cdots p_s^{a_s})-(g(1)h(p_1^{a_1}\cdots p_s^{a_s})+g(p_1^{a_1}\cdots p_s^{a_s})h(1))\}$$
$$=\ \{f(p_1^{a_1}\cdots p_s^{a_s})-f(p_1^{a_1})\cdots f(p_s^{a_s})\}\{k(p_1^{a_1}\cdots p_s^{a_s})-(g(1)h(p_1^{a_1}\cdots p_s^{a_s})+g(p_1^{a_1}\cdots p_s^{a_s})h(1))\}.$$

Since k is discriminative, (3.5) holds for $s\in\mathbb{N}$. Hence, f is multiplicative.
The proof of Part 3 is similar to that of Part 2.

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