The cut locus of Riemannian manifolds: a surface of revolution

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Abstract

This article reviews the structure theorems of the cut locus for very familiar surfaces of revolution. Some properties of the cut locus of a point of a Riemannian manifold are also discussed.

Keywords: Riemannian manifolds, surface of revolution, homeomorphic

1. Definition of a cut point and the cut locus

Let $\gamma:[0,a]\to M$ denote a minimal geodesic segment emanating from a point p on a complete connected Riemannian manifold M. The end point γ (a) is called a cut point of p along the minimal geodesic segment γ if any geodesic extension $\tilde{\gamma}:[0,b]\to M$, where b > a, of γ is not minimal anymore.

Definition 1.1 The cut locus C_p of a point p is the set of all cut points of p along minimal geodesic segments emanating from p.

It is very difficult to determine the structure of the cut locus of a point in a Riemannian manifold. The cut locus for a smooth surface is not a graph anymore, although it was proved by Myers in [1] and [2] that the cut locus of a point in a compact real analytic surface is a finite graph. In fact, Gluck and Singer [3] proved that there exists a 2-sphere of revolution admitting a cut locus with infinitely many branches. Their result implies that one cannot improve the following Theorem 1.3 without any additional assumption.

Theorem 1.2 [3] There exists a 2-sphere of revolution with positive Gaussian curvature such that the cut locus of a point admits an infinitely many branches.

Hebda proved in [4] that the distance function ρ to the cut locus of a point (in a complete 2-dimensional Riemannian manifold) is absolutely continuous where ρ is finite. Hence, for any pair of cut points of a point p can be connected by a rectifiable curve in C_p if the pair is in the same connected component.

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Theorem 1.3 [4] The cut locus of a point of a complete 2-dimensional (smooth) Riemannian manifold is a local tree, and the distance function to the cut locus is absolutely continuous where ρ is finite. In particular, the cut locus has a natural interior metric.

Remark 1.4 A topological space T is called a tree if for any two points p, q in T can be joined by a unique continuous curve. A topological space X is called a local tree if for any point $x \in X$ and any neighborhood V of x there exists a neighborhood $U \subset V$ of x which is a tree.

Remark 1.5 Hartman [5] studied detail differentiable structures of the cut locus of a simply closed smooth curve in a complete Riemannian manifold homeomorphic to Euclidean plane. His work was generalized to a simply closed curve in a 2-dimensional Riemannian manifold [6-8].

The cut locus of a point in a smooth Reimannian manifold cannot be a fractal set, i.e., the Hausdorff dimension is an integer see [9].

2. A surface of revolution homeomorphic toEuclidean plane

Definition 2.1 A complete Riemannian manifold (M, g) homeomorphic to Euclidean plane is called a surface of revolution if the manifold admits a point *p*, with $C_p = \phi$, such that the Riemannian metric g is expressed as

$$g = dr^2 + m(r)^2 d\theta^2$$

by making use of geodesic polar coordinates (r, θ) around *p*. The point *p* is called the vertex of the manifold.

It is known that a complete Riemannian manifold M homeomorphic to Euclidean plane is a surface of revolution with vertex p if and only if for each t > 0 the Gaussian curvature G is constant on S_p (t) := {q ∈ M |d(p, q) = t}.

Definition 2.2 A complete Riemannian manifold homeomorphic to Euclidean plane is called a von Mangoldt surface of revolution if the manifold admits a point *p* such that for any pair of points x, y with d(p, x) ≥ d(p, y), G(y) ≥G(x) holds. Here G denotes the Gaussian curvature of *M*.

Remark 2.3 A von Mangoldt surface of revolution is actually a surface of revolution, and a surface of revolution with vertex p is a von Mangoldt surface of revolution if and only if the Gaussian curvature is decreasing along each meridian, which means a geodesic emanating from the vertex *p*.

Typical examples of a von Mangoldt surface of revolution are paraboloids and 2-sheeted hyperboloids. Elerath [10] determined the structure of the cut locus for special classical surfaces of revolution.

Theorem 2.4 Let M(f) denote a surface of revolution defined by $z = f(\sqrt{x^2 + y^2})$, where f : R → (0,∞) denotes a smooth even function. If the Gaussian curvature is decreasing along each meridian, then for each point q of M (f), the cut locus C_q of *q* is empty or a subset of the meridian opposite to q.

Remark 2.5 The Sturm comparison theorem is a key tool in the proof of Theorem 2.4. Typical examples of a von Mangoldt surface are paraboloids $(z = a(x^2 + y^2))$ and a connected component of 2-sheeted hyperboloids $(z = a\sqrt{x^2 + y^2 + 1})$

Theorem 2.6 Let $(M, dr^2 + m(r)^2 d\theta^2)$ be a von Mangoldt surface of revolution with vertex p. Then the cut locus of a point q in M is either empty or a subset of the meridian opposite to *q*. More precisely, either $C_q = \phi$ or there exists a positive number t0 satisfying $C_p = \{(r,\theta) \mid \mathrm{r} \geq \mathrm{t0}, \theta = \pi + \theta(\mathrm{q})\}$

Definition 2.7 A point q of a surface of revolution $(M, dr^2 + d\theta^2)$ homeomorphic to Euclidean plane is called a *pole* if ex $p_q : T_q\ M \to M$ is injective (or equivalently $C_q = \phi$.

It is trivial that the vertex *p* of a surface of revolution is a pole. It is known that the set of poles of a surface of revolution forms a closed ball centered at the vertex and furthermore, we obtain

Theorem 2.8 Let $(M, dr^2 + d\theta^2)$ denote a surface of revolution with vertex p. Then the set of poles on M equals a closed ball centered at p and M admits a non-trivial pole if and only if

lim $inf_{r\to\infty} m(r)$ is non-zero and $\int_1^\infty m(r)^{-2}$ dr is *finite.*

Example 2.9 For a paraboloid of revolution, $\lim_{r\to\infty} m(r) = \int_1^\infty m(r)^{-2} dr = \infty$ Thus the vertex is a unique pole.

Example 2.10 For a 2-sheeted hyperboloid of revolution, $\lim_{r\to\infty} m(r) = \infty$ and $\int_1^\infty m(r)^{-2} dr$ is finite. Hence any point sufficiently close to the vertex is a pole.

3. A surface of revolution homeomorphic to a 2-sphere

Definition 3.1 A Riemannian manifold (M, g) homeomorphic to a 2-sphere is called a 2-sphere of revolution if M admits a point p with a single cut point q such that the Riemannian metric g is expressed as $g = dr^2 + m(r)^2 d\theta^2$ on M \ {p, q} by using geodesic polar coordinates (r, θ) around p. The point p and it's unique cut point is called a pair of poles of the 2-sphere.

Theorem 3.2 Let $(M, dr^2 + m(r)^2 d\theta^2)$ denote a 2-sphere of revolution with a pair of poles p; q satisfying the following two properties.

(3.1) M is symmetric with respect to the reflection fixing the equator r = 1/2•d (p, q).

(3.2) The Gaussian curvature of M is decreasing along a meridian from the point p to the point on the equator.

Then the cut locus of a point $x \in M \setminus \{p, q\}$, *with* $\theta(x) = 0$ is either a subarc of the open half opposite meridian $\theta^{-1}(\pi)$ to x or a single point on the open half opposite meridian. Moreover, if the cut locus of x is a single point, then the Gaussian curvature of *M* is constant.

Remark 3.3 A meridian of M means a periodic geodesic passing through *p* and *q*. For example, $\theta^{-1}(0) \cup \theta^{-1}(\pi) \cup \{p, q\}$ is a meridian.

A typical 2-sphere of revolution satisfying (1.1) and (1.2) is an ellipsoid de-fined by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, (0 < a < b)$$

Theorem 3.4 Let $(M, dr^2 + m(r)^2 d\theta^2)$ denote a 2-sphere of revolution with a pair of poles *p, q* satisfying (1.1) such that

(3.3) the Gaussian curvature of *M* is increasing along a meridian from the point *p* to the point on the equator.

Then, the cut locus of a point $x \in M \setminus \{p, q\}$ is either a single point or a subarc of the antipodal parallel $r = d(p, q) - r(x)$ to *x*. Moreover, if the cut locus of *x* is a single point, then the Gaussian curvature of *M* is constant.

A typical example of a 2-sphere of revolution satisfying (3.1) and (3.3) is an ellipsoid defined by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, (0 < a < b)$$

The structure of the cut locus of a general ellipsoid, i.e., a surface defined by $x^2 / a^2 + y^2 / b^2 + z^2 / c^2 = 1$, where 0 < a < b < c, has been determined by Itoh-Kiyohara [11]. The cut locus of a generic point of the ellipsoid is an arc. In [12] and [13], this result was generalized to a Liouville surface and Liouville manifolds.

Open Problem Let $(M, dr^2 + m(r)^2 d\theta^2)$ denote a 2-sphere of revolution with a pair of poles p, q. Suppose that a point $x \in M \setminus \{p, q\}$ *is a pole, i.e.,* C_x is a single point. Then, is any point $y \in M$ with $d(p, y) \leq d(p, x)$ a pole?

Remark 3.5 This would be true if M satisfies (3.1), and m is strictly increasing on (0, m(x)). The first claim of Theorem 2.8 is a non-compact version of this problem. From Theorem 3.4, it follows that the cut locus of a point on the equator $r = 1/2 \cdot d(p, q)$ is a subset of the equator. This theorem was generalized to a wider class of 2-spheres of revolution by Bonnard-Caillau-Sinclair –Tanaka [1].

Theorem 3.6 Let $(M, dr^2 + m(r)^2 d\theta^2)$ denote a 2-sphere of revolution satisfying (3.1). Suppose that the cut locus of a point on the equator $r = 1/2 \cdot d(p, q)$ is a subset of the equator. Then, the cut locus of a point $x \in M \setminus \{p, q\}$ is either a subarc of the antipodal parallel $r = d(p,q) - r(x)$ to x or a single point on the antipodal parallel.

Remark 3.7 Theorems 3.4 and 3.6 were generalized for a class of cylinders of revolution by *P*. Chitsakul [14], and [15] respectively.

Example 3.8 There exists a family $\{M_\lambda\}_\lambda$ of 2-spheres of revolution satisfying both properties in Theorem 3.6, but the Gaussian curvature is not mono-tonic along the meridian. By using geodesic polar coordinates *(r, θ)* around a point p of the unit sphere $S^2(1)$ we give a family of Riemannian metrics

$g_\lambda := dr^2 + m\lambda(r)^2 d\theta^2$, where $(\lambda \geq 0)$ is a parameter, on the unit sphere. Here $m_k := \sqrt{\lambda+1}\sin r / \sqrt{1+\lambda\cos^2 r}$. Then $M_k := (S^2(1), dr^r + m_\lambda^2(r)d\theta^2)$

satisfies both properties in Theorem 3.6, but the Gaussian curvature is not monotonic along a meridian if $\lambda > 2$.

4. A surface of revolution homeomorphic to a 2-torus

Let M be a standard torus in 3-dimensional Euclidean space defined by

$$(\sqrt{x^2 + y^2 - R})^2 + z^2 = r^2 (R > r > 0) .$$

The surface M is given by rotating the *(x, z)*-plane curve $\{(x,0,z) / (x - R^2 + z^2 = r^2)\}$ around the *z*-axis.

This surface has the following two properties. (4.4) It is symmetric with respect to the *(x, y)*-plane, i.e., it has a reflective symmetry with respect to the plane.

(4.5) The Gaussian curvature is increasing from the point $(R - r, 0, 0)$ to the point $(R + r, 0, 0)$ along the meridian defined by $y = 0$.

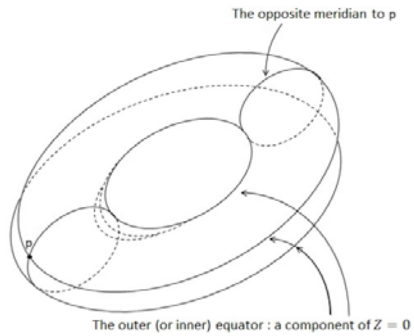
The structure of the cut locus for this torus is topologically complicated (see Figure 1). If we state it roughly,

Theorem 4.1 A cut point of a point $p = (x_0, 0, z_0)$, $x_0 > 0$, on the torus is a point on the meridian $\{(x, (x,0, z) \in M \mid x < 0\}$ opposite to p, a point on the antipodal parallel $\{(x, y, z) \in M \mid z = -z_0\}$, or a point on a (piecewise C1) Jordan curve which intersects the meridian opposite to p at a single point and is freely homotopic to each parallel.

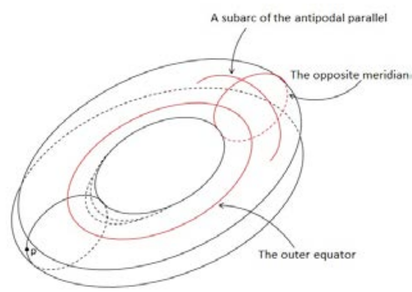
Remark 4.2 The structure of the cut locus is determined for a class of 2-torus of revolution which contains all standard tori in Euclidean space [16]. More precisely, let $(S^1 \times S^1, dt^2 + m(t)^2 d\theta^2)$ denote a torus with warped product Riemannian metric $dt^2 + m(t)^2 d\theta^2$, where dt^2 and $d\theta^2$ denote the Riemannian metric of a circle with length 2a and 2b respectively and m denotes a positive smooth warping function on *R* satisfying the following two properties:

(4.4) $m(-t) = m(t) = m(t + 2a)$ for any real number t.

(4.5) The Gaussian curvature $-\frac{m''}{m}(t)$ is increasing on $[0, a]$.



(a)



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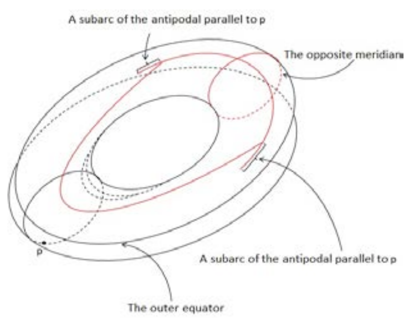

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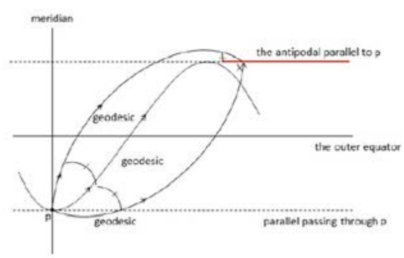

(d)

Figure 1. The structure of the cut locus

References

[1] Myers, S. B., **1935.** Connections between differential geometry and topology. I. Simply connected surfaces, *Duke Mathematics Journal*, 1, 376-391.

[2] Myers, S. B., **1936.** Connections between differential geometry and topology. II. Closed surfaces, *Duke Mathematics Journal*, 2(1), 95-102.

[3] Gluck, H. and Singer, D., **1979.** Scattering of geodesic fields. II, *Ann.Math.*110, 205-225.

[4] Hebda, J., **1994**. Metric structure of cut loci in surfaces and Ambrose's problem, *Journal of Differential Geometry*, 40, 621-642.

[5] Hartman, P., **1964.** Geodesic parallel coordinates in the large, *Amer. J. math.* 86, 705-727.

[6] Shiohama, K., and Tanaka, M., **1993.** The length function of geodesic parallelcircles, in Progress in Differential Geometry, *The Mathematical Society of Japan*, Tokyo, 299-308.

[7] Shiohama, K. and Tanaka, M., **1996.** Cut loci and distance spheres on Alexandrov surfaces, Acte de la Table Ronde de Géometrie Différentielle (Luminy, (1992), 531-559, Sémin. Congr., 1, *Société mathématique de France*, Paris.

[8] Shiohama, K., Shioya, T. and Tanaka, M., **2003.** The geometry of total curvature on complete open surfaces, Cambridge Tracts in Mathematics, 159. Cambridge University Press, Cambridge.

[9] Itoh, J.I. and Tanaka, M., **1998.** The dimension of a cut locus on a smooth Riemannian manifold, *Tohoku Mathematics Journal*, 50, 574-575.

[10] Elerath, D., **1980.** An improved Toponogov comparison theorem for non-negatively curved manifolds, *Journal of Differential Geometry*, 15, 187-216.

[11] Itoh, J. and Kiyohara, K., **2004.** The cut and the conjugate loci on ellipsoids, *Manuscripta Matematics*, 114, 247-264.

[12] Itoh, J. and Kiyohara, K., **2010.** The cut loci on ellipsoids and certain Liouville manifolds, *Asian Journal of Matematics*, 14, 257-289.

[13] Itoh, J. and Kiyohara, K., **2011.** Cut loci and conjugate loci on Liouville surfaces, *Manuscripta Mathematics*, 136, 115-141.

[14] Chitsakul, P., **2014.** The structure theorem for the cut locus of a certain class of cylinders of revolution I. *Tokyo Journal Mathematics*, 37, 473-484

[15] Chitsakul, P., **2015.** The structure theorem for the cut locus of a certain class of cylinders of revolution II. *Tokyo Journal Mathematics*, 38, 239-248

[16] Gravesen, J., Markvorsen, S., Robert, S. and Tanaka, M., **2005.** The cut locus of a torus of revolution, *Asian Journal of Mathematics*, 9, 103-120.