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Local Stability Analysis of Mathematical Model for Hemorrhagic Conjunctivitis Disease

S. Sangsawang, T. Tanutpanit, W. Mumtong and P. Pongsumpun*

Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

Abstract

Conjunctivitis is inflammation of the conjunctiva. It is commonly caused by bacteria and viruses. Allergic Conjunctivitis is distinguished by severe itching and allergen exposure. It is most commonly due to an infection or an allergic reaction. In this study, we analyzed the model for the spread of Hemorrhagic Conjunctivitis. Hemorrhagic Conjunctivitis caused by some bacteria and viruses. It can spread easily from person to person. In this model, we separate the human into susceptible, latent, infectious and recovered classes. We find the conditions for the existence of endemic and disease free equilibrium states. Also, basic reproductive number (R_0), the number of secondary cases that one infected individual will cause through the duration of the infectious period, is defined and shown for qualitative behavior of the model. Numerical simulations of the model are shown by solving a system of differential equations. It shows that the endemic and disease free equilibrium are unique and stable.

Keywords: Mathematical model, Hemorrhagic Conjunctivitis disease, equilibrium states, basic reproductive number.

1. Introduction

Conjunctivitis is a symptom which the white eye has red more than usual or the eye has the red eye more than the other side. It can be caused by virus, bacteria, allergy or chemical irritation. In this paper, we are interested in Conjunctivitis caused by virus or known as Acute Hemorrhagic Conjunctivitis (AHC). It is a highly communicable disease of tropical countries [1]. It occurs in the eye that found frequently in Thailand. It can be found all of the year and periodic outbreaks, especially during the rainy season because of the rain caused wetness and the virus is easily developed. Acute Hemorrhagic Conjunctivitis is simple and fast contact. This disease is caused by virus in *Adenovirus* group and *Enterovirus* group. Transmission occurs primarily via person to person contact or contact with contaminated objects by tears of patients attached with fingers and transmit from finger to eye directly. There is no contact from gaze, the air and eating together.

*Corresponding author: Tel: 662-329-8000 ext. 6196 Fax. 662-329-8400 ext. 283
E-mail: kppuntan@kmitl.ac.th

Each susceptible individual infect the virus and used the incubation period 1-2 days. The period of infection is about 14 days. People who have been infected with the virus have symptoms: red eyes, slightly swollen eyelids, irritating to the eyes, sore throat and not the gum in the eyes. Sometimes, there is the blood from the white eye and the lymph glands. Abnormalities of this disease are located at only eyelids and conjunctiva while part of black eye is normal so it does not affect for viewing. Because this disease is caused by virus, therefore no drug can cure this disease specificially [1-3]. In 2003, G. Chowell, E. Shim, F. Brauer, P. Diaz-Duenas, J. M. Hyman and C. Castillo-Chavez considered the model of an outbreak of Acute Hemorrhagic Conjunctivitis in Colima, Mexico in 2003. They use a simple model in order to estimate epidemiological and control parameters to reduce the spread of this disease [4].

In this paper, we study the transmission of Acute Hemorrhagic Conjunctivitis through mathematical modeling. We used the standard dynamical modeling method for analyzing the behavior of solutions and finding the method for decreasing the outbreak of this disease. From the data of Acute Hemorrhagic Conjunctivitis cases in Thailand from 2002 to 2011, we found that there were sporadic outbreak of this disease. This disease is usually found from June to December. It is shown in Figure 1.

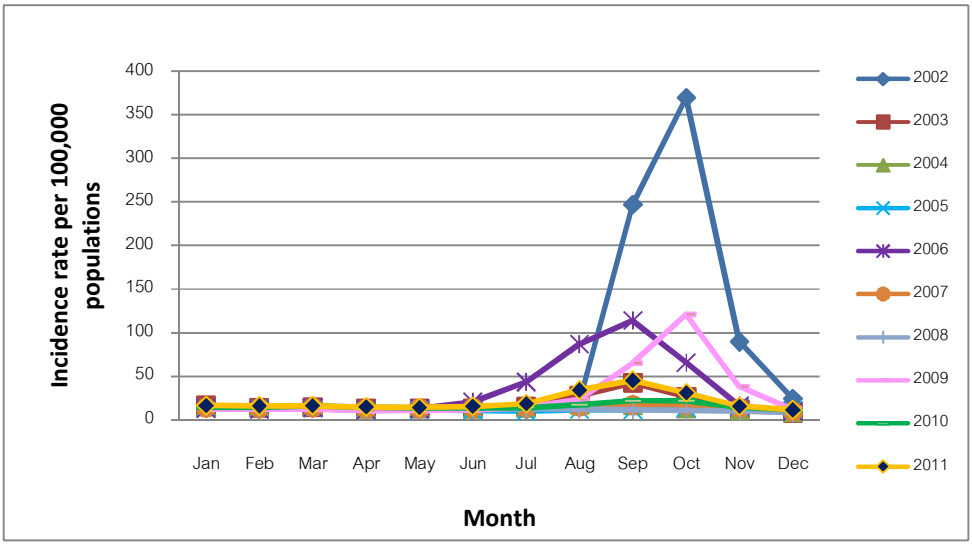


Figure 1 Reported cases of Acute Hemorrhagic Conjunctivitis per 100,000 populations in Thailand, year 2002-2011 [5-6].

This article is organized as follows: the formulation of model is presented in section 2. The numerical results and discussion are presented in section 3. Finally, Conclusion of our model is presented in section 4.

2. Transmission Model

We consider the transmission of Acute Hemorrhagic Conjunctivitis between people. The mathematical model of Acute Hemorrhagic Conjunctivitis is constructed by dividing the human population into four classes; susceptible, exposed, infected and recovered populations. The diagram of the transmission is presented in Figure 2.

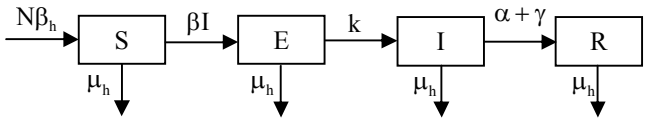


Figure2 Flow chart of the model.

We define

$S(t)$ is the number of susceptible human at time t,

$E(t)$ is the number of exposed human at time t,

$I(t)$ is the number of infectious human at time t,

$R(t)$ is the number of recovered human at time t,

and dynamical equation for each human group can be described as follows:

$$\frac{d}{dt}S(t) = N\beta_h - \mu_h S(t) - \beta I(t)S(t) \tag{1.1}$$

$$\frac{d}{dt}E(t) = \beta I(t)S(t) - \mu_h E(t) - kE(t) \tag{1.2}$$

$$\frac{d}{dt}I(t) = kE(t) - \mu_h I(t) - (\alpha+\gamma)I(t) \tag{1.3}$$

$$\frac{d}{dt}R(t) = (\alpha+\gamma)I(t) - \mu_h R(t) \tag{1.4}$$

with the conditions: $N(t) = S(t) + E(t) + I(t) + R(t)$

where

N = the total human population,

β_h = the birth rate of human populations,

μ_h = the death rate of human populations,

β = the transmission rate of Acute Hemorrhagic Conjunctivitis from person to person,

k = The incubation rate of Acute Hemorrhagic Conjunctivitis in human population,

α = The incidence rate who go to the doctor and get well,

γ = The incidence rate who don't go to the doctor but get well.

The total size of population is assumed to be constant. Thus, the rate of change for each human group equals to zero. This gives birth and death rates are equivalent for the human poputions $(\beta_h = \mu_h)$.

We normalize (1.1) - (1.4) by letting

$$S' = \frac{S}{N},\ E' = \frac{E}{N},\ I' = \frac{I}{N},\ R' = \frac{R}{N}$$

These give

$$\frac{d}{dt}S'(t) = \mu_h - \mu_h S'(t) - \beta_h N I'(t)S'(t) \tag{2.1}$$

$$\frac{d}{dt}E'(t) = \beta N I'(t)S'(t) - \mu_h E'(t) - kE'(t) \tag{2.2}$$

$$\frac{d}{dt}I'(t) = kE'(t) - \mu_h I'(t) - (\alpha+\gamma)I'(t) \tag{2.3}$$

where $R'(t)$ can be obtained from condition $S'(t)+E'(t)+I'(t)+R'(t)=1$.

3. Analysis of The Mathematical Model

3.1 Analytical Solution

The steady state (S^*, E^*, I^*) are found from setting the right hand side of (2.1) - (2.3) equal to zero. We obtain two steady state as follows:

1) The disease free steady state $E_0(1,0,0)$ and

2) The endemic disease steady state $E_1(S^*, E^*, I^*)$

where $S^* = \dfrac{\mu_h}{\mu_h + \beta N I^*}$, (3.1)

$$E^* = \frac{\mu_h \beta N I^*}{(\mu_h + k)(\mu_h + \beta N I^*)}, \tag{3.2}$$

$$I^* = \frac{k\mu_h}{(\mu_h + \alpha + \gamma)(\mu_h + k)} - \frac{\mu_h}{\beta N}. \tag{3.3}$$

The local stability of each steady state is defined by signs of all eigenvalues. The eigenvalues (λ) are the solutions of the characteristic equation;

$$\left| J_{E_i} - \lambda I_3 \right| = 0; \qquad i = 1,2$$

where J_{E_i} is the Jacobian matrix at the steady state E_i; i = 1,2 and I is the identity matrix dimension 3×3.

If all eigenvalues for each steady state have negative real parts, then that steady state is local stability.

i) The disease free steady state $E_0(1,0,0)$.

Jacobian matrix at the disease free steady state is given as follows:

$$J_{E_0} = \begin{pmatrix} -\mu_h & 0 & -\beta N \\ 0 & -\mu_h - k & \beta N \\ 0 & k & -\mu_h - \alpha - \gamma \end{pmatrix}_{(1,0,0)}$$

The characteristic equation of the above Jacobian matrix is

$$\left| J_{E_0} - \lambda I_3 \right| = 0$$

$$\lambda^3 + a_3\lambda^2 + a_2\lambda + a_1 = 0 \tag{4}$$

where

$$a_3 = 3\mu_h + k + \alpha + \gamma \tag{5}$$

$$a_2 = \mu_h(3\mu_h + 2(k + \alpha + \gamma)) + k(\alpha + \gamma) - k\beta N \tag{6}$$

$$a_1 = \mu_h(\mu_h + k)(\mu_h + \alpha + \gamma) - \mu_h \beta N k \tag{7}$$

These eigenvalue are negative when the coefficients a_3, a_2 and a_1 satisfy the Routh-Hurwitz criteria [7].

i) $a_3 > 0$ (8)

ii) $a_1 > 0$ (9)

iii) $a_3 a_2 - a_1 > 0$ (10)

We can see $a_3 = 3\mu_h + k + \alpha + \gamma$ is always positive, then we consider the next conditions. Consider a_1 is positive when

$$\beta Nk < (\mu_h + k)(\mu_h + \alpha + \gamma)$$

and $a_3 a_2 - a_1$ is positive when

$$\mu_h(4\mu_h + 2(k + \alpha + \gamma)) + k(\alpha + \gamma) > k\beta N.$$

Then three conditions satisfied following Routh-Hurwitz conditions. We deduce that E_0 is a locally asymptotically stable when $R_0 < 1$ ($R_0 = \dfrac{\beta Nk}{(\mu_h + \alpha + \gamma)(\mu_h + k)}$).

ii) The endemic disease steady state $E_1(S^*, E^*, I^*)$.

Jacobian matrix at the endemic disease steady state is given as follows:

$$J_{E_1} = \begin{pmatrix} -\mu_h - \beta NI^* & 0 & -\beta NS^* \\ \beta NI^* & -\mu_h - k & \beta NS^* \\ 0 & k & -\mu_h - (\alpha + \gamma) \end{pmatrix}_{(S^*, E^*, I^*)}$$

The characteristic equation of the above Jacobian matrix is

$$\left| J_{E_1} - \lambda I_3 \right| = 0$$

$$\lambda^3 + b_3\lambda^2 + b_2\lambda + b_1 = 0 \tag{11}$$

where

$$b_3 = 3\mu_h + \alpha + \gamma + \beta NI^* + k \tag{12}$$

$$b_2 = (\mu_h + \alpha + \gamma)(2\mu_h + \beta NI^* + k) + (\mu_h + \beta NI^*)(\mu_h + k) - k\beta NS^* \tag{13}$$

$$b_1 = (\mu_h + k)(\mu_h + \alpha + \gamma)(\mu_h + \beta NI^*) - \mu_h k\beta NS^* \tag{14}$$

These eigenvalue are negative when the coefficients b_3, b_2 and b_1 satisfy the Routh-Hurwitz criteria.

i) $b_3 > 0$ (15)

ii) $b_1 > 0$ (16)

iii) $b_3 b_2 - b_1 > 0$ (17)

We can see $b_3 = 3\mu_h + \alpha + \gamma + \beta NI^* + k$ is always positive, then we consider the next conditions. Consider b_1 is positive when

$$(\mu_h + k)(\mu_h + \alpha + \gamma)(\mu_h + \beta NI^*) > \mu_h k\beta NS^*$$

and $b_3 b_2 - b_1$ is positive when

$$(2\mu_h + \beta NI^* + k)((3\mu_h + \beta NI^* + k + \alpha + \gamma)(\mu_h + \alpha + \gamma) + (\mu_h + \beta NI^*)(\mu_h + k)) > k\beta NS^*(2\mu_h + \beta NI^* + k + \alpha + \gamma).$$

Then three conditions satisfied following Routh-Hurwitz conditions. We deduce that E_1 is a locally asymptotically stable when $R_0 > 1$ ($R_0 = \dfrac{\beta Nk}{(\mu_h + \alpha + \gamma)(\mu_h + k)}$).

3.2 Numerical Simulation

We consider the symptom of the patients who be infected with Acute Hemorrhagic Conjunctivites. In the simulations, the parameter $\mu_h = 1/(65\times 365)$ satisfy the real lifeexpectancy of 65 years for human. The other parameters are arbitrarily chosen.

The mathematical model which is analyzed in this study, the human population are assumed to have constant size. The quantity R_0 is the basic reproductive number of the disease where $R_0 = \dfrac{k\beta N}{(\mu_h+\alpha+\gamma)(\mu_h+k)}$, which is the number of secondary cases that one infected individual will cause through the duration of the infectious period. It means that every successive generation will diminish in size until its number approaches zero when $R_0 < 1$, while a disease to be capable of invading and establishing itself in a host person when $R_0 > 1$. From Figure 3 and Figure 4, it shows that the proportions of populations converge to the disease free steady state and oscillate to the endemic disease steady state for $R_0 < 1$ and $R_0 > 1$, respectively.

In Figure 5, we compare the time series solutions for the proportions of all populations when the basic reproductive numbers are different. It shows that the time of convergence for all populations are smaller when the basic reproductive number is higher, this means that one case can produce the greater number of secondary cases, and then the period of oscillation is shorter

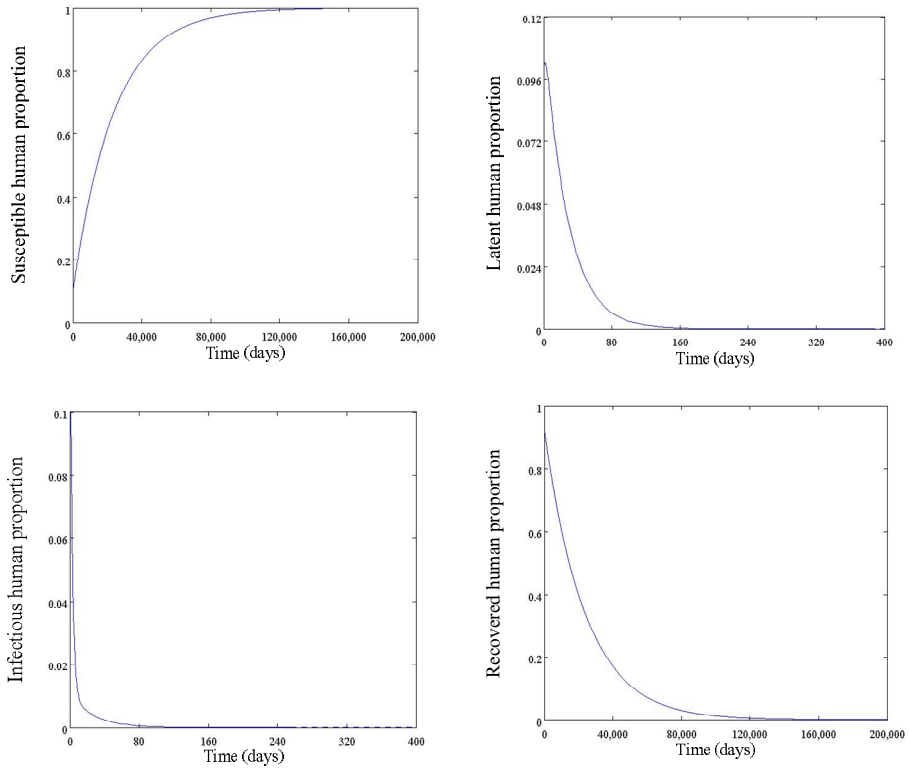


Figure 3 Numerical solutions of (2.1)-(2.3), demonstrate the times series of each human population, for $R_0 < 1$, $N = 100$, $\beta_h = 0.00004215$, $\mu_h = 0.00004215$, $\beta = 0.004$, $k = 0.04$, $\alpha = 0.08$, $\gamma = 0.33$ and $R_0 = 0.975$.

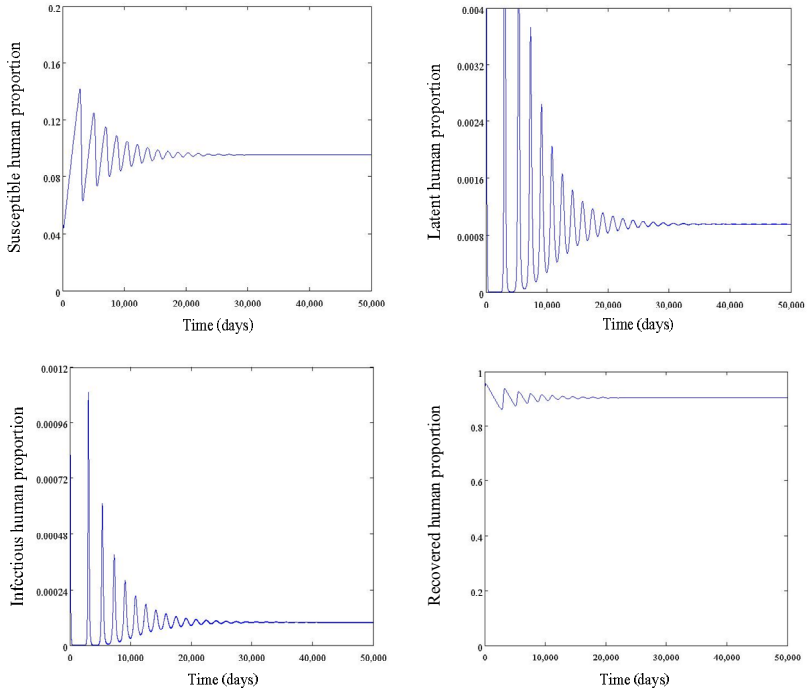
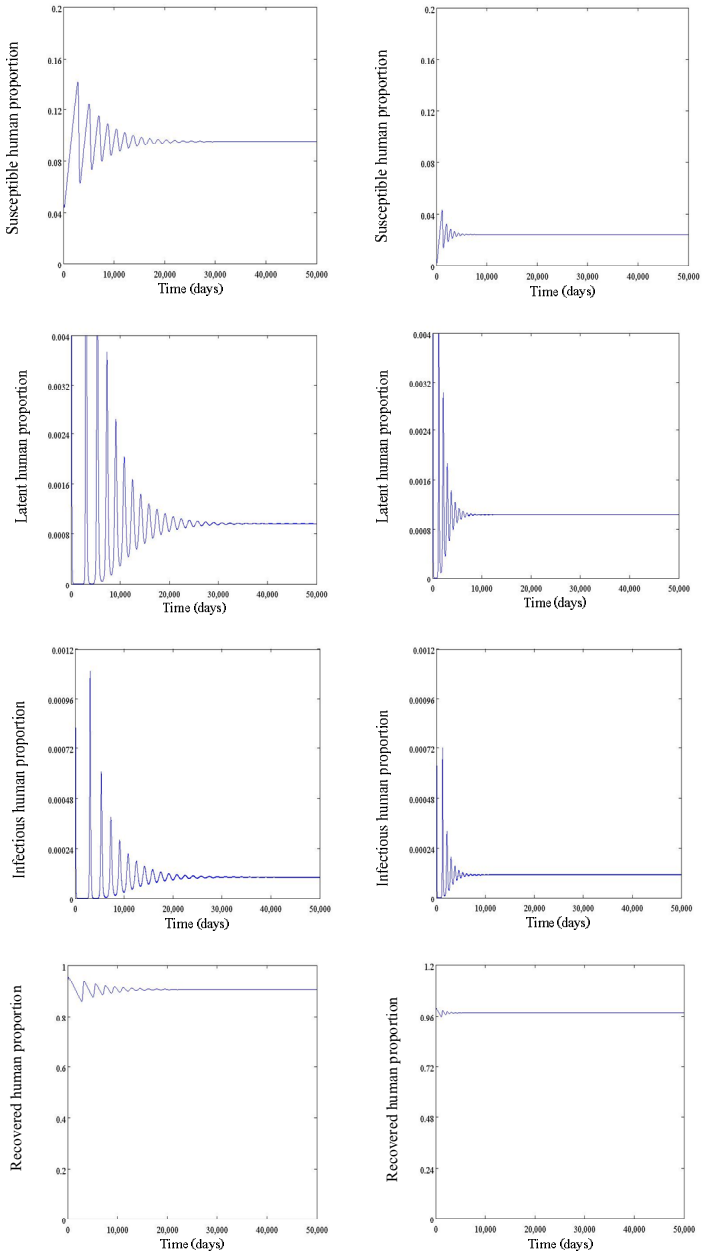


Figure 4 Numerical solutions of (2.1)-(2.3), demonstrate the times series of each human population, for $R_0 > 1$, $N = 1000$, $\beta_h = 0.00004215$, $\mu_h = 0.00004215$, $\beta = 0.004$, $k = 0.04$, $\alpha = 0.08$, $\gamma = 0.3$ and $R_0 = 10.5$.

The proportion of populations oscillate to the endemic disease state E_1 where $E_1 : (S^*, E^*, I^*, R^*) = (0.0951, 0.00095, 0.9038, 0.0001)$.



5a) 5b)

Figure 5 Numerical solutions of (2.1)-(2.3), demonstrate the times series solutions of each human population.

a) For R_0 =10.5, N =1000, $\beta_h = 0.00004215$, $\mu_h = 0.00004215$, $\beta = 0.004$, $k = 0.04$, $\alpha = 0.08$ and $\gamma = 0.3$.

b) For R_0 =42, N =4000, $\beta_h = 0.00004215$, $\mu_h = 0.00004215$, $\beta = 0.004$, $k = 0.04$, $\alpha = 0.08$ and $\gamma = 0.3$.

4. Conclusions

In conclusion, if we can control the reproductive number as a value less than 1, then this disease will be eradicated. Therefore the control program of the transmission would have to be done continuously in order to reduce the transmission of the disease.

5. Acknowledgements

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