

A Survey of Extremal Rays for Fano Manifolds

Toru Tsukioka*

Department of Mathematics, Faculty of Science, Tokai University, Kanagawa, Japan

Abstract

In this expository note, we give an introduction to algebraic geometry with the emphasis on Fano manifolds which forms one of the building blocks of algebraic varieties. After a brief review on the theory of intersection numbers, we introduce numerical invariants on Fano manifolds and discuss related problems.

Keywords: algebraic varieties, Fano manifolds, extremal ray

1. Introduction

Algebraic geometry is a branch of mathematics studying the structures of algebraic varieties i.e. the set of zeros of a system of polynomial equations. It is natural to consider such varieties not in an affine space (=euclidian n -space) but in a projective space.

Let k be a field. Recall that the n -dimensional projective space $P^n(k)$ is defined as the quotient of the set

$$k^{n+1} - \{(0, 0, \dots, 0)\}$$

by the equivalence relation:

$$(x_0, x_1, \dots, x_n) = \lambda (y_0, y_1, \dots, y_n)$$

where λ is some non-zero element of k . The ground field k is usually assumed to be algebraically closed. A simple reason is briefly explained in the book [9] Chapter I. Furthermore, if we consider the field of complex numbers $\mathbb{C}$, then the study of algebraic varieties is naturally linked to other branches of mathematics e.g. complex geometry, differential geometry or topology. In this note, we put $P^n = P^n(\mathbb{C})$. The points in P^n are written in the form of homogeneous coordinates: $(x_0 : x_1 : \dots : x_n)$ with $(x_0, x_1, \dots, x_n) \in k^{n+1} - \{(0, 0, \dots, 0)\}$. For example, in P^2 we have $(1 : i : -1) = (-i : 1 : i)$ since $(1, i, -1) = i \cdot (-i, 1, i)$ in $\mathbb{C}^3$. Consider the algebraic variety $X := \{(x:y:z:w) \in P^3 | xz - yw = 0\}$. The dimension of X is two, because it is defined by one equation in the three dimensional projective space. To investigate the structure of X , we consider the curve $C := \{(x:y:z:w) \in P^3 | x=y=0\}$ which is contained in X . We have the self-intersection number $(C^2) = 0$ (see [1] Chapter I for the intersection theory on algebraic surfaces). This does not happen for any curve on P^2 (any curve on P^2 has strictly positive self-intersection number). We conclude that X is a different algebraic variety from P^2 . However, X and P^2 are birational. Indeed, if we consider the projection map: $\varphi: P^3 \rightarrow P^2$ given by

$$(x : y : z : w) \rightarrow (x : y : z),$$

*Corresponding author: Tel: 0463-58-1211 Fax: 0463-58-9543
E-mail: tsukioka@tokai-u.jp

then the restriction to X is a birational map. This X is called quadric surface because it is defined by an equation of degree two. The main problem of algebraic geometry is the birational classification of algebraic varieties (see [6] Chapter 1.8). For example, P^2 and the X defined above are in a same equivalence class in this point of view. It is known that the (smooth) cubic surface is also birational to P^2 . However, a surface defined by equations of degree 4 or more are not birational to P^2 and hence are in different classes (this can be verified by using numerical invariants e.g. "Kodaira dimension" (see [7])). In higher dimensions, the birational classification is very complicated. However, there is a method to get a "simple" model of algebraic varieties in each birational class (see [4]).

2. Fano manifolds

A Fano manifold is a projective variety whose anticanonical bundle is ample (see [5] Chapter 1 for a definition of "ample"). The anticanonical bundle of a projective space is expressed as

$$-K = (n+1)H$$

where H is a hyperplane in P^n . Since H is (very) ample, we conclude that P^n is a Fano manifold. Let X be a hypersurface of degree d in P^{n+1} , then, by "adjunction formula" we have

$$-K_X = (-K - X)|_X = ((n+2)H - dH)|_X = (n+2-d)H|_X.$$

Hence X is a Fano manifold if and only if $n+2-d > 0$, i.e. $d \leq n+1$. From this example, we can say that (the degree of) Fano manifolds is bounded in some sense. This observation is correct in general: the anticanonical degree of the Fano manifolds of dimension n is bounded by a function of n . This bound implies that there are only finitely many types of Fano manifolds in each dimension. This very deep result has been shown using the theory of rational connectedness (see [3] p.251 for a brief history)

The classification of Fano manifolds up to dimension three is following (see [8] and references therein):

- ($\dim X = 1$) X is isomorphic to P^1
- ($\dim X = 2$) these are "Del Pezzo surfaces": P^2 , $P^1 \times P^1$, the blow-ups of P^2 at most 8 general points
- ($\dim X = 3$) there are 109 deformation types of smooth Fano 3-folds.

In the dimension greater than or equal to four, there are only partial results on explicit classifications of Fano manifolds. The study of Fano manifolds is very important in algebraic geometry. According to the minimal model theory (MMP), every uniruled variety is birationally equivalent to a Fano fibration i.e. a fiber space whose fibers are Fano manifolds (e.g. the projection $P^n \times Y \rightarrow Y$ is a Fano fibration since the fiber P^n is a Fano manifold). Fano manifolds themselves have rich geometry: they are of great interest concerning the rationality problem of algebraic varieties, the theory of algebraic groups and the Einstein metric in differential geometry etc. (see [10]).

3. Cone of curves and extremal rays for Fano manifolds

Let X be a (smooth) projective variety of dimension n . A divisor D on X is a formal sum of codimension one subvarieties with integral (rational or real) coefficients. A 1-cycle C on X is a formal sum of 1-dimensional subvarieties with integral (rational or real) coefficients. Divisors and 1-cycles are related by the intersection pairing:

$$(D, C) \mapsto D \cdot C$$

which defines a bilinear map. Remark: the intersection number $D \cdot C$ is a number of points in the intersection $D \cap C$ with multiplicity. For example, let X be the complex 3-space C^3 , D a hyperplane say $D := \{(x, y, z) \mid z = 0\}$ and C the curve (1-cycle) defined by $\{(x, y, z) \mid x = 0, z = y^2\}$.

Then, the intersection $D \cap C$ consists of one point $(0, 0, 0)$. However, we have $D \cdot C = 2$ because D has multiplicity two at the point $(0, 0, 0)$.

Two 1-cycles A and B are said to be numerically equivalent and denoted by $A \equiv B$ if we have

$$D \cdot A = D \cdot B$$

for any divisor D on X . We define the (real) vector space $N_1(X) := \{ \text{1-cycles on } X \} / \equiv$ whose dimension is known to be finite. Its dimension denoted by $\rho(X)$ is called Picard number. Note that for two classes $[A]$ and $[B]$ in $N_1(X)$, we have $[A] = [B]$ if and only if $A \equiv B$. There is a subcone called Kleiman-Mori cone or cone of curves:

$$NE(X) := \{ [C] \in N_1(X) \mid C: \text{1-cycle with positive coefficients} \}$$

which plays an important role in the minimal model theory. We often consider its closure $\overline{NE}(X)$ in euclidian topology of $N_1(X)$.

An extremal ray R of $NE(X)$ is a half line satisfying the following:

$$u, v \in NE(X) \text{ and } u + v \in R \implies u, v \in R.$$

It is known that the Kleiman-Mori cone of a Fano manifold is polyhedral and each extremal ray is generated by a rational curve (i.e. a curve birational to P^1). Let X be a Fano manifold. We consider the following two natural numbers:

- $a(X) :=$ the number of extremal rays of $NE(X)$
- $l(X) :=$ the minimum of $-K \cdot C$ where C is a rational curve in some extremal ray

By definition, we have $a(X) \geq \rho(X)$. Using a classification result from [11], we get the following:

Theorem Let X be a smooth Fano 4-fold having a birational contraction. Assume $l(X) \geq 2$. Then we have $a(X) = \rho(X)$.

4. Conclusions

In general, $a(X)$ is greater than the Picard number $\rho(X)$. If X is a cubic surface in P^3 , there are 27 lines contained in X . Let $C_1, C_2, \dots, C_{27}$ be these rational curves. By an elementary observation, we see that each $[C_i]$ generates an extremal ray of $NE(X)$. Therefore, we obtain $a(X) = 27$. On the other hand, the cubic surface is obtained by blowing up 6 points in P^2 . Hence we have $\rho(X) = 7$.

It is natural to ask the following:

Problem Find an explicit bound for the number of extremal rays for Fano manifolds.

Remark: The explicit upper bound for the Picard number of Fano manifolds of dimension n is conjectured to be $9n/2$ (for n even), $9(n-1)/2 + 1$ (for n odd). However, it is only verified up to dimension three (see [2]).

References

- [1] Beauville, A. **1983**. *Complex algebraic surfaces*. London Mathematical Society Student Texts 34, Cambridge University Press
- [2] Casagrande, C. **2012**. *On the Picard number of divisors in Fano manifolds*. The Annales Scientifiques de l'Ecole Normale Supérieure.
- [3] Kollar, J. **1996**. *Rational curves on algebraic varieties*. *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge. Band 32*, Springer
- [4] Kollar, J. and Mori, S. **1998**. *Birational geometry of algebraic varieties*. Cambridge tracts in Mathematics 134, Cambridge University Press
- [5] Lazarsfeld, R. **2004**. *Positivity in algebraic geometry I*. *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge. Band 48*, Springer
- [6] Hartshorne, R. **1977**. *Algebraic geometry*. Graduate Texts in Mathematics 52, Springer
- [7] Iitaka, S. **1972**. *Genus and classification of algebraic varieties. I*. *Sugaku* 24, 14-27

- [8] Parchin, A. and Shavarevich, I. **1999**. Algebraic geometry V. Encyclopaedia of Mathematical Sciences 47, Springer
- [9] Shavarevich, I. **1988**. Basic Algebraic Geometry 1. Springer.
- [10] Hwang, J.-M. **1998**. *Some Recent Topics in Fano Manifolds*. Trends in Mathematics Information Center for Mathematical Sciences 1, 26–30
- [11] Tsukioka, T. **2012**. *On the minimal length of extremal rays for Fano four-folds*. Math. Zeit. 271, 555–564