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Some Remarks on Visible Actions on Multiplicity-free Spaces

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Abstract

This paper provides a summary of the recent work on visible actions, based on Sasaki [1-3]. A holomorphic action of a Lie group G on a complex manifold D is called stronglyvisible if a real submanifold$S$ meeting every G-orbit in D and an anti-holomorphic diffeomorphism σ preserving each G-orbit such that$\sigma|_S = \mathrm{id}_S$ exists. In case of complex linear spaces, a deep relationship between visible actions and multiplicity-freerepresentations is found. A holomorphic representation of a complex reductive it's Lie group has a strongly visible action if and only if its polynomial ring is multiplicity-free as it's representation. Furthermore, the dimension of our choice of a slice coincides with the rank of the semigroup of highest weights occurring in the polynomial ring.

Keywords: Visible actions, slice, multiplicity-free representation, rank

1. Introduction

A holomorphic action of a Lie group G on a connected complex manifold D is called *strongly visible* if a real submanifold$S$ and an anti-holomorphic diffeomorphism σ on D exists and the following conditions are satisfied [4, Definition 3.3.1].

(V.1) $D' \coloneqq G \cdot S$ is open in D,

(S.1) $\sigma|_S = \mathrm{id}_S$,

(S.2) $\sigma(v) \in G \cdot x$ for any $x \in D'$.

We say that the above S is a *slice* for the strongly visible G-action on D. We note that the slice S is automatically totally real, namely, $T_x S \cap J_x(T_x S) = \{0\}$for any $x \in S$ (see [4, Remark 3.3.2]). Here, J denotes the complex structure on$D$.

Strongly visible actions are visible actions in the sense that there exists a totally real submanifold$S$ satisfying (V.1) and $J_x(T_x S) \subset T_x(G \cdot x)$ for any $x \in S$ (see [4, Theorem 4]). The geometry of strongly visible actions deals with complex manifolds having infinitely many orbits.

One of the simplest examples is the standard action of a torus $T \coloneqq \{z \in \mathbb{C}: |z| = 1\}$on $\mathbb{C}$. Then, the generic orbits are circles with origin as their center, from which their dimension equals one. Let e be the standard basis of $\mathbb{C}$. This basis defines the standard real form$\mathbb{R}e$. We denote by σ the complex conjugation with respect to $\mathbb{R}e$. We set $S \coloneqq \mathbb{R}_+ e$. Then, the set $T \cdot S$ coincides with $\mathbb{C}^\times \coloneqq \mathbb{C} \setminus \{0\}$, which is open in $\mathbb{C}$. Clearly, σ satisfies$\sigma|_S = \mathrm{id}_S$. Moreover, the symmetry of circles with respect to diameters shows that σ preserves each T-orbit in $\mathbb{C}^\times$. Hence, the T-action on $\mathbb{C}$ is strongly visible.

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The notion of (strongly) visible actions is the key geometric setting for propagation theorem of multiplicity-free property. For a G-equivariant Hermitian holomorphic vector bundle over D with strongly visible action, the multiplicity-free property propagates from fibers to the space of holomorphic sections. This theory gives a unified explanation of the multiplicity-free theorems including both finite-dimensional and infinite-dimensional case (see [4, 5]).

Recently, we found various examples of strongly visible actions in connection with multiplicity-free representations. The paper [4] provides a number of strongly visible actions. Furthermore, Hermitian symmetric spaces and flag varieties of Type A are studied by visible actions. In our previous study [6], we considered $\mathbb{C}^{\times}$-bundles over the complexification of non-tube type Hermitian symmetric spaces. Now, the study of non-symmetric spherical homogeneous spaces is in progress (see [7, 8]).

This is a survey paper on the study of strongly visible actions on linear spaces. Then, we omit a proof of our theorems, which related to recent papers [1, 2] and the preceding paper [3]. The aim of this paper is to illustrate our study by giving some examples.

2. Preliminaries

In this section, we review the basic notion and structural results on multiplicity-free spaces, related to References [9-11].

Let V be a finite-dimensional complex vector space and $G_{\mathbb{C}}$ a connected complex reductive Lie group. Suppose that we are given a holomorphic representation of $G_{\mathbb{C}}$ on V. We say that $(G_{\mathbb{C}}, V)$ is a *multiplicity-free space* if the induced representation of $G_{\mathbb{C}}$ on the polynomial ring $\mathbb{C}[V]$defined by

$$f(v) \mapsto \pi(g)f(v) = f(g^{-1} \cdot v)$$

is multiplicity-free. Multiplicity-free spaces were classified by Kac [11, Theorem 3] under the assumption that the $G_{\mathbb{C}}$-action on V is irreducible, and by Benson and Ratcliff [9, Theorem 2] and Leahy [12, Theorem 2.5], independently, where the $G_{\mathbb{C}}$-action on $V$is not irreducible.

Suppose that $(G_{\mathbb{C}}, V)$ is a multiplicity-free space. By definition, the polynomial ring $\mathbb{C}[V]$ is the multiplicity-free direct sum of irreducible representations of $G_{\mathbb{C}}$. By the Cartan—Weyl highest weight theory, irreducible finite-dimensional holomorphic representations of $G_{\mathbb{C}}$ are parameterized by the highest weights. We denote by ρ_λ those representations with highest weight λ. Then, we write

$$\mathbb{C}[V] \simeq \textstyle\bigoplus_{\lambda \in \Lambda} \rho_\lambda \,.$$

Here, it is known that the support Λ is finitely generated by linearly independent highest weights$\lambda_1, \dots, \lambda_r$. The rank of multiplicity-free space $(G_{\mathbb{C}}, V)$, denoted by $\mathrm{rank}(G_{\mathbb{C}}, V)$, is defined as the rank of Λ, namely, the number r of the generators $\lambda_1, \dots, \lambda_r$.

Explicit irreducible decomposition of $\mathbb{C}[V]$, in particular, the rank of $(G_{\mathbb{C}}, V)$ was found by Howe and Umeda [13] for irreducible case and by Benson and Ratcliff [9] and Knop [14] for reducible case.

3. Main results

Retain the setting of Section 2. Let G_u be a compact real form of $G_{\mathbb{C}}$. Due to the Weyl's unitary trick, the category of holomorphic representations of $G_{\mathbb{C}}$is equivalent to that of complex representations on G_u. In particular, the $G_{\mathbb{C}}$-action on V is still holomorphic. Our main theorem is stated as follows:

Theorem 3.1 ([1, 2]).The following two conditions for a linear action on V are equivalent:

(i) $(G_{\mathbb{C}}, V)$ is a multiplicity-free space.

(ii) The G_u-action on V is strongly visible.

Theorem 3.1 concludes that the classification of strongly visible linear actions coincides with that of multiplicity-free spaces.

The implication (i) ⇒ (ii) is a special case of propagation theorem of multiplicity-free property (see [4, 5]). We prove the opposite implication (ii) ⇒ (i) by finding a concrete description of slice S and anti-holomorphic diffeomorphism σ for each multiplicity-free space $(G_{\mathbb{C}}, V)$ according to the classification due to Kac, Benson—Ratcliff and Leahy.

Thanks to our choice of slice S and anti-holomorphic diffeomorphism σ in the proof of the implication (ii) ⇒ (i), we find the following theorem:

Theorem 3.2 ([1, 2]). For any strongly visible G_u-action on a multiplicity-free space $(G_{\mathbb{C}}, V)$, we can find a slice S, anti-holomorphic diffeomorphism σ on V such that the following conditions are satisfied:

(a) $\dim S = \operatorname{rank}(G_{\mathbb{C}}, V)$.

(b) σis involutive, namely, $\sigma^2 = \mathrm{id}$.

We give a proof of Theorems 3.1 and 3.2 simultaneously dividing into two papers [1, 2], one of which considers the case where the G_u-action on V is irreducible, the other the reducible case.

As an application of the study of strongly visible actions on linear spaces, we give a necessary and sufficient condition for a nilpotent orbit in a complex simple Lie algebra to have a strongly visible action, see the future paper [15]. In our study, the following theorem plays a useful role to the induction theorem of strong visibility via momentum map.

Theorem 3.3 ([3]).Retain the setting of Theorem 3.2. Then, there exists an anti-holomorphic involutiveautomorphism$\sigma_{\#}$ on $G_{\mathbb{C}}$ which stabilizes G_u such that

(c) $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}^{\sigma_{\#}} = \operatorname{rank} \mathfrak{g}$.

(d) $\sigma_{\#}$is a compatible automorphism with σ for the strongly visible $G_{\mathbb{C}}$-action on$V$ namely,

$$\sigma(g \cdot v) = \sigma_{\#}(g) \cdot \sigma(v) (g \in G_{\mathbb{C}}, v \in V) \quad \cdots\cdots\cdots (3.1)$$

Here, the Lie algebra $\mathfrak{g}^{\sigma_{\#}}$ satisfying the condition (c) is called a *normal* real form of$\mathfrak{g}$. For complex semisimple Lie algebra$\mathfrak{g}$, normal real forms are unique up to isomorphism. For instance, $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{o}(n, n)$are normal real forms of $\mathfrak{gl}(n, \mathbb{C})and\mathfrak{o}(n, \mathbb{C})$, respectively.

We note that the conditions (V.1), (S.1), and (3.1) of Theorem 3.2 for $(S, \sigma, \sigma_{\#})$shows (S.2). In fact, let $v = g \cdot s$ be an element of$G_u \cdot S$. Then, we have

$$\sigma(v) = \sigma(g \cdot s) = \sigma_{\#}(g) \cdot \sigma(s) = \sigma_{\#}(g) \cdot s = \sigma_{\#}(g) g^{-1} \cdot v \in G_u \cdot v.$$

A proof of Theorem 3.3 will be given in the preceding paper [3].

4. Example of visible actions on linear spaces

The rest of this paper is devoted to giving a typical example of Theorems 3.1—3.3 as follows.

Let $G_{\mathbb{C}}$ be the direct product of two general linear groups $G_{\mathbb{C}} = GL(m, \mathbb{C}) \times GL(n, \mathbb{C})$ and V a vector space $V = M(m, n; \mathbb{C})$ consisting of complex $m \times n$-matrices. Without loss of generality, weassume $m \leq n$. Let $G_{\mathbb{C}}$ act on V by

$$(g, h) \cdot X = gXh^{-1}\big((g, h) \in G_{\mathbb{C}}, X \in V\big).$$

Then, this action is linear and holomorphic. The polynomial ring $\mathbb{C}[V]$decomposes into the multiplicity-free sum of irreducible representations of $G_{\mathbb{C}}$as follows:

$$\mathbb{C}[M(m, n; \mathbb{C})] \simeq \bigoplus_{\lambda_1 \geq \cdots \geq \lambda_m \geq 0} \pi^{GL_m}_{(-\lambda_m, \ldots, -\lambda_1)} \otimes \pi^{GL_n}_{(\lambda_1, \ldots, \lambda_m, 0, \ldots, 0)} \quad \cdots\cdots\cdots (4.1).$$

Here, $\pi^{GL_m}_{\lambda}$denotes the irreducible representation of $GL(m, \mathbb{C})$ with highestweightλ $\in \mathbb{Z}^m$, and so on. The irreducible decomposition (4.1) is well-known as the (GL_m, GL_n)-duality.

Our geometric viewpoint of strongly visible actions is explained as follows. Let G_u be a compact real form of$G_{\mathbb{C}}$. Then, $G_u \simeq U(m) \times U(n)$. Here, we take S to be the set of diagonal matrices whose entries are all zero, namely,

$$S = \left\{ \left(\begin{array}{ccc|c} r_1 & & & \\ & \ddots & & O \\ & & r_m & \end{array} \right) : r_1, \dots, r_m \in \mathbb{R} \right\}$$

The well-known fact that the complex matrices are diagonalizable by the action of unitary groups shows that we have$V = G_u \cdot S$. We define an anti-holomorphic diffeomorphismσ on V by the standard complex conjugation, namely,

$$\sigma(X) = \overline{X}\big(X \in M(m, n; \mathbb{C})\big),$$

and an anti-holomorphic involution $\sigma_\#$ on $G_{\mathbb{C}}$ by

$$\sigma_\#(g, h) = (\overline{g}, \overline{h})\big((g, h) \in G_{\mathbb{C}}\big).$$

Clearly, $\sigma|_S = \mathrm{id}_S$ and $G_{\mathbb{C}}^{\sigma_\#} = GL(m, \mathbb{R}) \times GL(n, \mathbb{R})$. Furthermore, the direct computation shows that we have $\sigma\big((g, h) \cdot X\big) = \sigma_\#(g, h) \cdot \sigma(X)$for $(g, h) \in G_{\mathbb{C}}$ and $X \in V$. Therefore, the G_u-action on V is strongly visible.

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