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Contributed Paper

A Smaller Cover of the Moser's Worm Problem

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Abstract

The Moser's worm problem asks for a smallest set on the plane that contains a congruent copy of every unit arc. Such smallest covering set has not been found yet. The smallest known cover constructed by Norwood and Poole in 2003 [6] has area 0.260437. In this work, we adapt their idea to construct a smaller cover of area 0.26007. We also simplify the proof that the set constructed this way contains a congruent copy of every unit arc.

Keywords: covering unit arcs, worm problem

1. Introduction

In 1966, Leo Moser posted a famous problem asking for the smallest set on the plane that contains a congruent copy of every unit arc [1]. The problem is later known as the Moser's worm problem. This problem is similar to another well known problem, the Lebesgue's universal cover problem that asks for the smallest set on the plane that contains a congruent copy of every set with diameter at most 1. Due to their similarity, both problems are still open. Unfortunately, the existence of the solution of the worm problem is not yet confirmed. However, if we insist the solution set to be convex, the Blaschke's selection theorem will guarantee the existence.

A direct application of the Moser's worm problem is to economize the cost of the material required to cover a variation of objects with a certain length. For instance, a company can design a bandage shaped similar to a Moser's worm cover to guarantee that the bandage can cover a one-inch cut of any shape.

For convenience, we call a set that contains a congruent copy of every set in a family F, a **cover** for F. The simplest cover for unit arcs is a disk of radius 1. To cover a unit arc γ by this disk, we just locate the center of the disk at one end of γ. Then γ will be contained in the disk which has area $\pi \approx 3.14159$. By placing the midpoint of γ at the center, we can see easily that a disk of radius $\frac{1}{2}$ is also a cover. The smaller disk has area $\frac{\pi}{4} \approx 0.78540$. Despite the difficulty of the problem, many covers are found and keep getting smaller and smaller.

The first few notable covers are found in 1970's. The first cover is by Meir [2] where the cover is a half of the disk of radius $\frac{1}{2}$ and hence of area $\frac{\pi}{8} \approx 0.39270$. Then Wetzel [2] showed that a sector of area 0.34501 is a cover. After that, Gerriets [3], and then Gerriets and Poole [4] found smaller covers of areas 0.3214 and 0.28610 respectively. In 1989, Norwood, Poole, and Laidacker constructed a cover of area 2.7524 [5]. All these covers are convex so far. And finally, in 2003, Norwood and Poole came up with a non-convex cover of area 0.260437 [6]. See Figure 1.

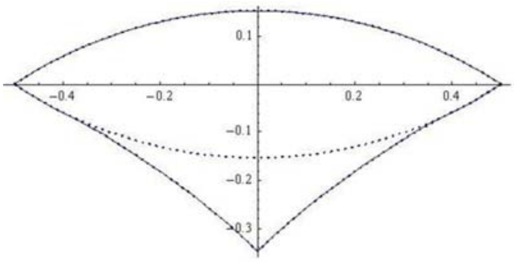

Figure 1. The non-convex cover by Norwood and Poole of area 0.260437.

The upper boundary of this cover is a circular arc and the lower boundary is composed of 2 parabolic arcs between 2 circular arcs.

In this work, we improve the latest cover in [6] by replacing the upper boundary with an elliptic arc and follow the same construction concept. We also simplify the proof in [6] while maintain the same numberings for the properties and cases.

Beside the original problem by Moser, many variations of the problem has been investigated. For example, the problem of finding the smallest cover for closed unit arcs. Another interesting variation is to find the smallest cover for convex unit arcs. For the latter, the currently smallest cover found is by Wichiramala [7].

2. MATERIALS AND METHODS

2.1 The New Cover

Our new cover C^+ in Figure 2 has area 0.26007, which is smaller than 0.260437 of the current smallest cover by Norwood and Poole [6].

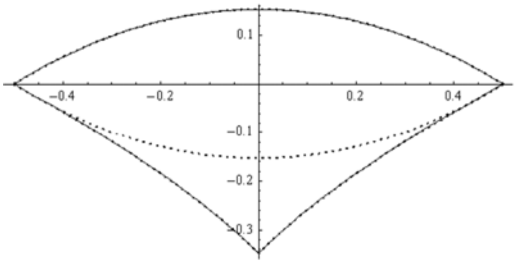

Figure 2. The new cover of area 0.26007.

To construct C^+, we start with an ellipse E: $(\frac{x}{a})^2 + (\frac{y - y_c}{b})^2 = 1$ where E contains $(\frac{1}{2}, 0)$ as in Figure 3.

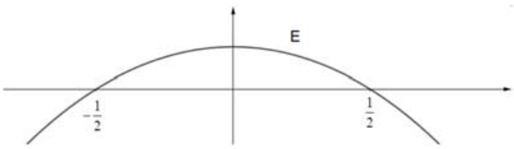

Figure 3. The ellipse E.

By simple calculation, we have $y_c = -b\sqrt{1 - \frac{1}{4a^2}}$. As we consider the part of E on the upper half plane, we may regard E as the graph of $f(x) = b\sqrt{1 - (\frac{x}{a})^2} + y_c$. Next we let L be the locus of points whose distances to Y-axis and to E add up to $\frac{1}{2}$ (see Figure 4).

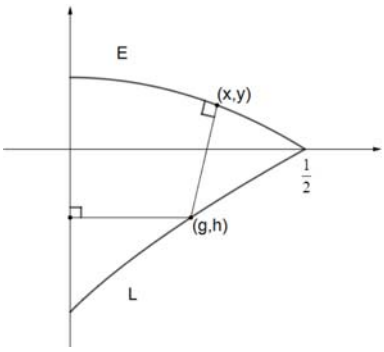

Figure 4. The locus L.

Hence $g + \sqrt{(g - x)^2 + (h - y)^2} = \frac{1}{2}$ and the slope s of the segment from (g, h) to (x, y) is $\frac{y - h}{x - g} = \frac{-1}{f'(x)}$.

Thus $g + (x - g)\sqrt{1 + s^2} = \frac{1}{2}$, and then $g = x - \frac{x - \frac{1}{2}}{1 - \sqrt{1 + s^2}}$. By a similar argument on the slope, we get $h = y - (x - g)s$.

From now on, we may treat g and h as the functions of x. Let l be the function with graph L. Hence $l(x) = h(g^{-1}(x))$.

Our proposed new cover C^+ is bounded by the graphs $y = f(x)$ (the top of C^+) and $y = \min(l(x), -f(x))$ (the bottom of C^+).

2.2 Properties of the New Cover

From the construction of the bottom of C^+, we have the following property.

Property A: Every arc from Y-axis that meet the bottom of C^+ and then meet the top of C^+ must be at least $\frac{1}{2}$ in length.

As C^+ is parametrized by a and b, we wish to find a and b that minimize the area of C^+ with the following property.

Property B: Every arc from (0, 0) that meets the top of C^+ and then meets the bottom of C^+ must be at least $\frac{1}{2}$ in length.

We use Mathematica's Minimize command to show that any arc from (0, 0) that meets the top of the cover C^+ with $a = 1.95272$, $b = 4.58588$ and then meets the bottom of C^+ has length at least 0.500001. This cover has area 0.26007. Therefore, this cover satisfies the Property B and has area smaller than the cover constructed by Norwood and Poole in [6].

These two properties will help to prove that C^+ can cover any planar arc of length one.

3. RESULTS AND DISCUSSION

In this section, we will prove that C^+ is a cover for every unit arc. Note that, as we will follow the similar idea of proof as in [6], we keep the same numbering of cases in the proof of the main theorem.

Theorem 3.1 Every unit arc can be covered by C^+.

Proof. We assume the contrary that an arc cannot be covered by C^+. By [8], we may assume that γ is simple, i.e. it does not intersect itself. Let m be the midpoint of γ where it is divided into 2 subarcs α and β of length $\frac{1}{2}$.

Our placing scheme is to place γ into C^+ by locating m on the Y-axis and move γ down until it touches the bottom of C^+. Our moving scheme is also to keep m on the Y-axis. For convenience, we use T^+ and B^+ to denote the top and the bottom of C^+ respectively.

We divide into cases according to whether each half of γ can touch B^+.

Case 1: Only β can touch B^+. We consider the situation where m is as lowest as possible. We call this property MPS1 (minimal positioning scheme). We then subdivide this case into smaller cases according to whether β is in C^+.

Case 1A: β is not in C^+.

By properties A and B, the y-coordinate of is above X-axis (see Figure 5).

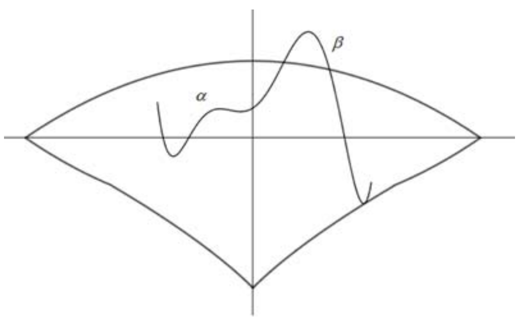

Figure 5. β touches B^+ and is not in C^+.

Now let C^{++} be the reflected copy of C^{++} as illustrated in Figure 6, and denote the reflect copy of T^{+} and B^{+} by T^{++} and B^{++} respectively.

Due to its isometry to C^{+}, we have that β is not above T^{++}. By properties A and B applying to C^{++}, β does not cross B^{++}. Hence we may translate γ upward (if necessary) so that γ touchesB^{++}. Since we are in Case 1, α never touches B^{++} and thus we now place γ with m closer to B^{++} than to B^{+} at the original position, a contradiction to MPS1.

Case 1B: β is in C^{+}. Hence α is not in C^{+}

Let $\hat{a}$ be a points on α above T^{+} (see Figure 7).

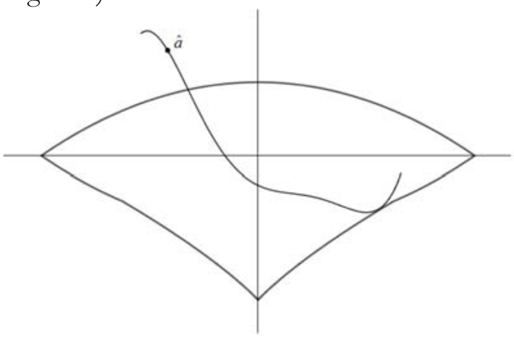

Figure 7. β touches B^{+} and is in C^{+}.

Note that both γ and β are under B^{++}. Now translate γ upward until it touches B^{++}. We will divide into subcases according to whether or not β is in C^{++}.

Subcase 1B1: β is in C^{++}. Thus α is not in C^{++}. In particular, α has a point $\hat{b}$ below B^{++} (see Figure 8).

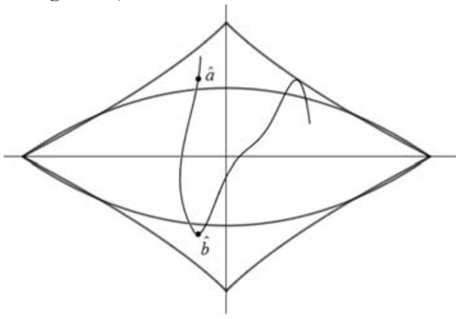

Figure 8. β touches B^{++} and α is not in C^{++}.

In this position, $\hat{a}$ is still above T^{+}. Now go back to the original position of γ and translate C^{++} downward to touch γ. We have that β is between T^{+} and T^{++}, and is on one side of the subarc from $\hat{a}$ to $\hat{b}$ (see the shaded region in Figure 9).

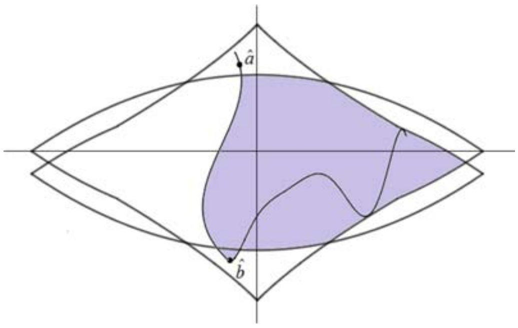

Figure 9. β is in the shaded region.

Then if we rotate γ (counter clockwise according to the figure) so that m is on Y-axis and a and b are on the same level. Without loss of generality, because the arc is simple, we may assume β is above the subarc from a to b. In this orientation, as we translate γ down, α will touch T^{+}, a contradiction to the assumption in case 1.

Subcase 1B2: β is not in C^{++} (see Figure 10).

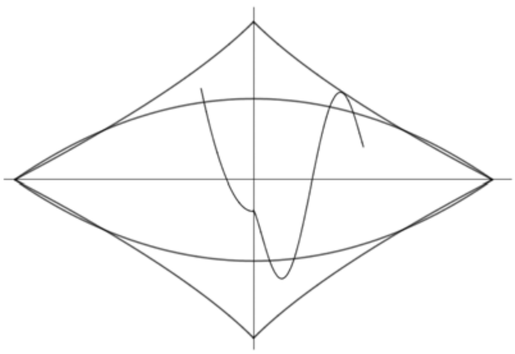

Figure 10. β touches B^{++} and is not in C^{++}.

From the original position of γ, translate C^{++} downward until T^{++} touches β.

Let S be the tangent line touching α and β from above (see Figure 11).

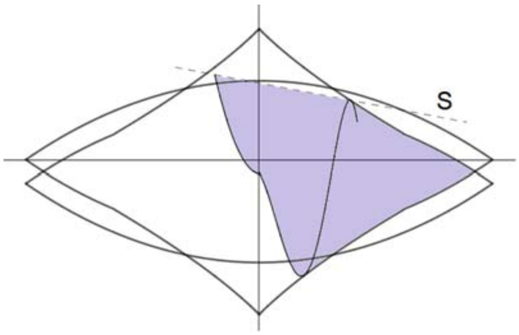

Figure 11. The line S touches α and β from above.

Now we move down and rotate γ and S together until has slope -1. In particular S is now parallel to the tangent line of the top right of B^{++} (see Figure 12).

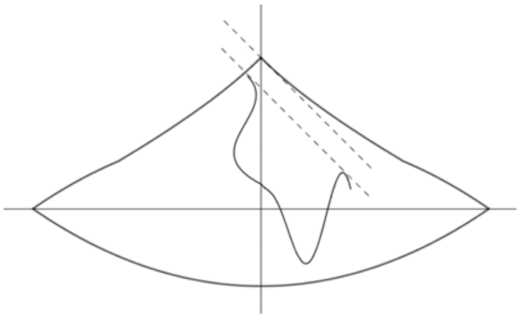

Figure 12. γ and S are rotated so that S has slope -1.

While translating γ up, since α cannot touch B^{++}, β will eventually touch the left half of B^{++}. Now we rotate γ counter clockwise while keeping β touching B^{++} until β also touches the right half of B^{++}. As α is above β, β is not in C^{++} and also travels from Y-axis to visit both the left and the right halves of B^{++}. This contradicts Properties A and B.

Case 2: both α and β may touch B^{+}. By continuity, there are orientations that both halves of γ touch B^{+}. We consider the situation that m is as low as possible and both halves touch B^{+}. We call this property MPS2.

Without loss of generality, we may suppose that β is not in C^{+} (see Figure 13). By properties A and B, $y_m > 0$.

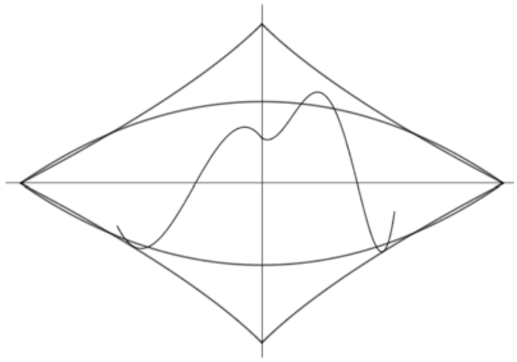

Figure 13. Both α and β touch B^{+}.

Hence both α and β do not cross B^{++}. When we translate γ up, if needed, either α or β touches B^{++}.

Case 2A: β touches B^{++} at $\hat{b}$. Hence β is in C^{++} and thus α is not in C^{++} and is not below B^{+}. Let $\hat{a}$ be a point of α below T^{++} as in Figure 14.

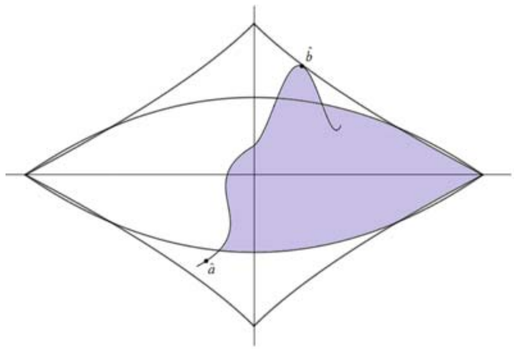

Figure 14. β touches B^{++} at $\hat{b}$ and a point $\hat{a}$ of α is below T^{++}.

Hence α is below B^{++} and then we may rotate γ clockwise while keeping β touching B^{++}. As the subarc from $\hat{a}$ to $\hat{b}$ is on the left of the remaining subarc (of β), the rotation will move α to touch B^{++}. Otherwise, β will also touch the left half of B^{++}, a contradiction to Properties A and B similar to the end of case 1B2.

During the rotation, if $y_m = 0$ when β touches B^{++} at b', we may use the fact that

$x_{b'} > \frac{1}{5}$ to conclude that further rotation clockwise would bring *m* up. This computation is due to the fact that the shortest distance from the origin to B^{++} is greater than 0.2663 (which can be confirmed computationally) and that the original β touches B^{+}. This is the other necessary condition used in the proof in [6]. Note, however, that the shortest distance in [6] is greater than 0.2645. Hence we reach a position that *m* is closer to B^{++} than to B^{+} at the original position, a contradiction to MPS2.

Case 2B: α touches B^{++}. Similar to the previous case, we rotate γ counterclockwise until β also touches B^{++}. During the rotation, if $y_m = 0$ when α touches B^{++} at c', further rotation would bring *m* up as $x_{c'} > \frac{1}{5}$.

This completes the proof.

4. Conclusions

In this work, we modify the cover of area 0.260437 constructed by Norwood and Poole to obtain a smaller cover of area 0.26007 provided that we use $a = 1.95272$ and $b = 4.58588$. Note that there are other values of *a* and *b* such that the corresponding cover contains all unit worms and has area smaller than 0.260437. The aforementioned values of *a* and *b* have yet to be proven to give the smallest cover among the family of covers constructed this way.

Acknowledgement

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